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Geometric and arithmetic realized comoments

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Abstract
The author investigates realized comoments that overcome the drawback of conventional ones and derive the following findings. First, the author proves that (even generalized) geometric implied lower-order comoments yield neither geometric realized third comoment nor fourth moment. This is in contrast to previous studies that produce geometric realized third moment and arithmetic realized higher-order moments through lower-order implied moments. Second, arithmetic realized joint cumulants are obtained through complete Bell polynomials of lower-order joint cumulants. This study's realized measures are unbiased estimators and they can, therefore, overcome the drawbacks of conventional realized measures.

Keywords Realized joint cumulants, Realized comoments, Log returns, Implied moments, Aggregation property
Paper type Research paper

1. Introduction

The framework suggested by Andersen *et al.* (2003) produces low-frequency variance from high-frequency returns. This so-called realized variance is defined as a sum of squares of sub-periodical returns. Kraus and Litzenberger (1976) and Dittmar (2002) demonstrate the relationship between higher-order moments and expected returns, and the concept of the realized variance has been extended to realized higher-order moments. In many studies, including those of Amaya *et al.* (2015), Sim (2016), Kim (2016), Mei *et al.* (2017), Kinateder and Papavassiliou (2019), and Ahmed and Al Mafrachi (2021) [1], a realized kth order moment is defined as a sum of kth orders of sub-periodical returns. However, according to Amaya *et al.* (2015) and Bae and Lee (2021), these conventional realized higher-order moments can reflect neither the volatility of volatility nor cross-period relation among sub-periodical returns and are, therefore, flawed. Several studies attempt to resolve these problems by providing unbiased realized moments, and such research is summarized in Table 1.

The revised realized moments are developed based on Neuberger's (2012) Aggregation Property, through which the author presents arithmetic and geometric realized third moments using changes in prices and implied variances [2]. Bae and Lee (2021) extend the arithmetic realized moments in two folds. One is the extension of moments to comoments, and the other is an extension of the order from three to four. Furthermore, Fukasawa and

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Matsushita (2021) provide arithmetic moments of general orders. However, to the best of the author's knowledge, geometric comoments, geometric moments above the third order and arithmetic comoments above the fourth order have not yet been developed.

The current study attempts to complete Table 1. Our first target is the geometric realized moments and comoments. Many financial studies use geometric returns (log-returns) because they have useful features such as time-additivity. Accordingly, Neuberger (2012) proposes geometric realized third moment. To find the missing geometric measures in the aforementioned table, we extend information set to include lower order moments because all the revised moments are obtained through the lower order moments. However, unlike the aforementioned studies, the current research demonstrates that (even generalized) implied variance and covariance do not yield realized third comoment, although they yield realized covariance. Moreover, we reveal that (even generalized) implied third moment does not yield realized fourth moments.

Our second target is the arithmetic realized comoments for general orders. We previously mentioned the usefulness of the log-returns, and as shown in Table 1, arithmetic realized comoments up to the fourth-order are developed. However, arithmetic returns are also as well-used as geometric returns, and financial studies require the estimation of higher-order comoments. For example, Rubinstein (1973) extends the traditional Capital Asset Pricing Model (CAPM)

$$E[r_i] = r_f + \lambda E[(r_M - E[r_M])(r_i - E[r_i])]$$

with $\lambda = \frac{1}{\sigma_M^2} E[r_M - r_f]$ to

$$E[r_i] = r_f + \sum_{l=2}^{\infty} \lambda_l E\left[(r_M - E[r_M])^{l-1}(r_i - E[r_i])\right],$$

and Chung *et al.* (2006) and Hung (2008) demonstrate that comoments above the fourth order are priced. Accordingly, we attempt to identify the realized comoment above the fourth-order under the arithmetic sense. To do so, we extend Fukasawa and Matsushita's (2021) arithmetic realized cumulants [3]. While Neuberger (2012) and Bae and Lee (2021) attempt to obtain all functions satisfying the Aggregation Property given information set, Fukasawa and Matsushita (2021) present a rule among realized cumulants. Adopting their methodology, we obtain arithmetic realized joint cumulants through complete Bell polynomials of lower-order joint cumulants. Our realized measures are unbiased estimators and they can, therefore, overcome the drawbacks of conventional realized measures.

The rest of the paper is organized as follows. Neuberger's (2012) Aggregation Property is reviewed, and generalized geometric moments are defined in section 2. The non-existence of geometric higher order moments and comoments is demonstrated in section 3. Joint

Table 1. Revised realized moments and comoments

Order	Arithmetic realized moments	Geometric realized moments
Panel A. Realized moments		
3	Neuberger (2012)	Neuberger (2012)
4	Bae and Lee (2021)	-
Above 4	Fukasawa and Matsushita (2021)	-
Panel B. Realized comoments		
3	Bae and Lee (2021)	-
4	Bae and Lee (2021)	-
Above 4	-	-

cumulants are explained and arithmetic realized joint cumulants outlined in section 4. Finally, concluding remarks are presented in section 5.

2. Preliminary: aggregation property and generalized geometric realized moments

Consider a martingale process S_t and a partition $\{t_0, t_1, \ldots, t_N\}$ on $[0, T]$ such that $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_N = T$. Equation (1) holds for $k = 2$.

$$E_0\left[(S_T - S_0)^k\right] = E_0\left[\sum_{j=1}^{N} \left(S_{t_j} - S_{t_{j-1}}\right)^k\right] \tag{1}$$

Owing to this relation, $\sum_{j=1}^{N}(S_{t_j} - S_{t_{j-1}})^2$ is referred to as realized second moment or realized variance. However, Equation (1) does not hold for the higher-order ($k \ge 3$), which makes obtaining realized higher-order moments non-straightforward. To solve this problem, Neuberger (2012) proposes the aggregation property that generalizes Equation (1) as follows.

Definition 2.1. Aggregation property

Let $X = (X_t, 0 \le t \le T)$ be an adapted vector-valued stochastic process defined on a filtration. A function g on a vector-valued process X satisfies the *AP* (*aggregation property*) if

$$E_r[g(X_u - X_r)] = E_r[g(X_u - X_t)] + E_r[g(X_t - X_r)], \forall(r, t, u)\ 0 \le r \le t \le u \le T. \tag{2}$$

Owing to the law of the iterated expectations, when a function g satisfies the AP, we have

$$E_0[g(X_T - X_0)] = E_0\left[\sum_{j=1}^{N} g\left(X_{t_j} - X_{t_{j-1}}\right)\right]. \tag{3}$$

In this regard, $\sum_{j=1}^{N} g(X_{t_j} - X_{t_{j-1}})$ can be called a realized $E_0[g(X_T - X_0)]$.

To develop the realized moments of log returns, X_t needs to contain log prices $s_t = \ln S_t$, and additional arguments can contribute to constructing the abundant functions that satisfy the AP. For example, Neuberger (2012) uses Δs and Δv^N that are changes in log price s_t and specific generalized variance v_t^N, respectively. Furthermore, he demonstrates that $e^{\Delta s} - 1$, Δs, Δv^N, $e^{\Delta s}(\Delta v^N + 2\Delta s)$, and their linear combination satisfy the AP when the stock price S_t is a martingale. Thus, the following form satisfies the AP.

$$\begin{aligned} g^N\left(\Delta s, \Delta v^N\right) &= -12\left(e^{\Delta s} - 1\right) + 6\Delta s - 3\Delta v^N + 3e^{\Delta s}\left(\Delta v^N + 2\Delta s\right) \\ &= 3\Delta v^N\left(e^{\Delta s} - 1\right) + 6\left(\Delta s e^{\Delta s} - 2e^{\Delta s} + \Delta s + 2\right) \end{aligned} \tag{4}$$

Moreover, the martingale property yields $E_t[g^N(\ln S_T - \ln S_t, v_T^N - v_t^N)] = E_t[K(\ln S_T - \ln S_t)]$ for $K(x) = 6(xe^x - 2e^x + x + 2) = x^3 + O(x^4)$. Thus, Neuberger (2012) refers to,

$$\sum_{j=1}^{N} g^N\left(\ln S_j - \ln S_{j-1}, v_{t_j}^N - v_{t_{j-1}}^N\right) \tag{5}$$

as a realized third moment of log return $\ln S_T - \ln S_0$. However, the study presents neither any realized comoments nor realized fourth moments. It may be resolved by additional information of their lower-order implied comoments of log returns. According to Neuberger (2012), implied variance contributes to constructing the realized third moment for both arithmetic and log returns. Similarly, Bae and Lee (2021) show that realized comoments for the arithmetic returns require lower-order moments and comoments. Thus, implied covariance and variances of log returns may contribute to the realized third comoment of log returns, and implied variance and third moment of log returns may contribute to the realized fourth moment of log returns.

To consider covariation, we use two martingale processes $S_{1,t}$ and $S_{2,t}$, and their log values are $s_{1,t}$ and $s_{2,t}$, respectively. For the variant functions satisfying the AP, we allow flexibility on the forms of implied comoments, and we define generalized comoments as follows [4].

Definition 2.2. (Implied) generalized (k, l)-comoment

We refer to $E_t[f^{k,l}(s_{1,T} - s_{1,t}, s_{2,T} - s_{2,t})]$ as a *generalized* (k, l)*-comoment* at time t when $f^{k,l}$ is an analytic function such that $\frac{f^{k,l}(a,b)}{a^k b^l} \to 1$ as $(a, b) \to (0, 0)$. For convenience, we call it a *generalized* ($k + l$)*-moment* and replace $f^{k,l}$ with f^{k+l} if k or l is zero.

Equipped with the above log prices processes and implied comoments, we investigate higher-order realized comoments. Consider a partitioned vector process $x = (s_1, s_2, m)$, where m is a vector process of comoments. When a function g satisfies

$$E_0\left[\sum_{j=1}^{N} g\left(x_{t_j} - x_{t_{j-1}}\right)\right] = E_0\left[g_r^{k,l}(s_{1,T} - s_{1,0}, s_{2,T} - s_{2,0})\right] \tag{6}$$

with a function $g_r^{k,l}(\cdot,\cdot)$ such that

$$g_r^{k,l}(s_{1,T} - s_{1,0}, s_{2,T} - s_{2,0}) \approx (s_{1,T} - s_{1,0})^k (s_{2,T} - s_{2,0})^l, \tag{7}$$

$E_0[\sum_{j=1}^{N} g(x_{t_j} - x_{t_{j-1}})]$ is close to ordinary comoment. Thus, a realized comoment is defined as follows.

Definition 2.3. Realized (k,l)-comoment

For a partitioned vector process $x = (s_1, s_2, m)$ including a vector process m_t, let us call

$$\sum_{j=1}^{N} g\left(x_{t_j} - x_{t_{j-1}}\right) \tag{8}$$

a *realized* (k, l)*-comoment* if a function g satisfies the AP and is decomposed as follows

$$g(x_\tau - x_t) = \phi(x_\tau - x_t) + g_r^{k,l}(s_{1,\tau} - s_{1,t}, s_{2,\tau} - s_{2,t}), \tag{9}$$

where ϕ is a function that satisfies $E_t[\phi(x_T - x_t)] = 0$, and $g_r^{k,l}$ is a function such that that $\frac{g_r^{k,l}(a,b)}{a^k b^l} \to 1$ as $(a, b) \to (0, 0)$. For convenience, we refer to Equation (8) as a *realized* ($k + l$)*-moment* if k or l is zero.

Note that when a function g satisfies the AP and has the decomposition in Equation (9), we have

$$E_0\left[\sum_{j=1}^{N} g\left(x_{t_j} - x_{t_{j-1}}\right)\right] = E_0[g(x_T - x_0)] = E_0\left[g_r^{k,l}(s_{1,T} - s_{1,0}, s_{2,T} - s_{2,0})\right]. \tag{10}$$

Thus, it is close to the standard comoment when $g_r^{k,l}(s_{1,T} - s_{1,0}, s_{2,T} - s_{2,0}) \approx (s_{1,T} - s_{1,0})^k\ (s_{2,T} - s_{2,0})^l$.

3. Nonexistence of geometric realized higher-order comoments

Based on Definition 2.2, we denote the generalized 2-moment for the asset $i \in \{1,2\}$ as v_i with its underlying function $f^2(\cdot)$. In addition, let us denote the generalized comoment as v_c with its underlying function $f^{1,1}(\cdot,\cdot)$. We first investigate the function satisfying the AP given the information set x that includes log prices (s_1, s_2), variances (v_1, v_2) and covariance (v_c) as follows.

Proposition 3.1. An analytic function g satisfies the AP on the vector valued process $x = (s_1, s_2, v_1, v_2, v_c)$ if and only if g is represented as follows:

$$\begin{aligned} g(\Delta s_1, \Delta s_2, \Delta v_1, \Delta v_2, \Delta v_c) = {} & h_1\left(e^{\Delta s_1}-1\right) + h_2\Delta s_1 + h_3\left(e^{\Delta s_2}-1\right) + h_4\Delta s_2 \\ & + h_5\Delta v_1 + h_6\Delta v_2 + h_7\Delta v_c + h_8(\Delta v_1 - 2\Delta s_1)^2 \\ & + h_9(\Delta v_2 - 2\Delta s_2)^2 + h_{10}(\Delta v_1 - 2\Delta s_1)(\Delta v_2 - 2\Delta s_2) \\ & + h_{11}e^{\Delta s_1}(2\Delta v_c - \Delta v_2 + 2\Delta s_2) + h_{12}e^{\Delta s_2}(2\Delta v_c - \Delta v_1 + 2\Delta s_1) \\ & + h_{13}e^{\Delta s_1}(\Delta v_1 + 2\Delta s_1) + h_{14}e^{\Delta s_2}(\Delta v_2 + 2\Delta s_2) \end{aligned} \tag{11}$$

for some constants $h_1, \cdots, h_{14}$, which satisfy one of the following five conditions:

(1) $h_{12} = h_{13} = h_{14} = 0, f^{1,1}(\Delta s_1, \Delta s_2) = \Delta s_2(e^{\Delta s_1} - 1)$ and $f^2(\Delta s) = 2(e^{\Delta s} - \Delta s - 1)$,

(2) $h_{11} = h_{13} = h_{14} = 0, f^{1,1}(\Delta s_1, \Delta s_2) = \Delta s_1(e^{\Delta s_2} - 1)$ and $f^2(\Delta s) = 2(e^{\Delta s} - \Delta s - 1)$,

(3) $h_{11} = h_{12} = h_{13} = h_{14} = 0$ and $f^2(\Delta s) = 2(e^{\Delta s} - \Delta s - 1)$,

(4) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = 0$ and $f^2(\Delta s) = 2(\Delta s e^{\Delta s} - \Delta s + 1)$,

(5) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = h_{13} = h_{14} = 0$.

The proof is provided in Appendix 1.

Proposition 3.1 uses the information of implied covariance v_c in addition to the variation of a single process in Neuberger (2012). It makes it possible to obtain new terms that satisfy the AP: the 10th term $(\Delta v_1 - 2\Delta s_1)(\Delta v_2 - 2\Delta s_2)$ with conditions (1), (2) and (3), the 11th term $e^{\Delta s_1}(2\Delta v_c - \Delta v_2 + 2\Delta s_2)$ with condition (1), and 12th term $e^{\Delta s_2}(2\Delta v_c - \Delta v_1 + 2\Delta s_1)$ with condition (2). These new terms are generalizations of $(\Delta v_1 - 2\Delta s_1)^2$ and $e^{\Delta s_1}(\Delta v_1 + 2\Delta s_1)$ observed in Neuberger (2012) in that the new terms become these when we set $S_{2,t}$ to be identical to $S_{1,t}$. The new terms may contribute to constructing new realized comoments, and Corollary 3.2 states the result.

Corollary 3.2. When the information set is given by $x = (s_1, s_2, v_1, v_2, v_c)$, there is not a realized (2,1)-comoment but a realized (1,1)-comoment.

The proof is in Appendix 1.

According to Corollary 3.2, we could not obtain realized (2,1)-comoment even when we have all its lower-order moment and comoment. This result is in contrast to Neuberger (2012) and Bae and Lee (2021), who obtain the realized third moment under both the arithmetic and log return and realized third comoment under the arithmetic return through their lower-order moments. Instead, Corollary 3.2 shows that Δv_c with Δv_1 or Δv_2 produces the realized (1,1) comoment through

$$\begin{aligned} g(\Delta s_1, \Delta s_2, \Delta v_2, \Delta v_c) &= -\Delta v_c + \frac{1}{2}\Delta v_2 + \frac{1}{2}e^{\Delta s_1}(2\Delta v_c - \Delta v_2 + 2\Delta s_2) - \Delta s_2 \\ &= \left(e^{\Delta s_1} - 1\right)\left(\Delta v_c - \frac{1}{2}\Delta v_2\right) + \left(e^{\Delta s_1} - 1\right)\Delta s_2, \end{aligned} \tag{12}$$

or

$$g(\Delta s_1, \Delta s_2, \Delta v_2, \Delta v_c) = \left(e^{\Delta s_2} - 1\right)\left(\Delta v_c - \frac{1}{2}\Delta v_1\right) + \left(e^{\Delta s_2} - 1\right)\Delta s_1. \tag{13}$$

Now, let us investigate functions satisfying the AP when the information includes higher-order moments for the single security. They are log price (s), implied second moment (m_2) and implied third moment (m_3), where the underlying function for the kth moment is denoted by $f^k(\cdot)$.

Proposition 3.3. An analytic function g on a vector valued process $x = (s, m_2, m_3)$ has the Aggregation Property on the vector valued process x if and only if g is represented as follows:

$$\begin{aligned} g(\Delta s, \Delta m_2, \Delta m_3) = {} & h_1\left(e^{\Delta s} - 1\right) + h_2\Delta s + h_3\Delta m_2 + h_4\Delta m_3 \\ & + h_5(\Delta m_2 + a\Delta m_3 - 2\Delta s)^2 + h_6(\Delta m_2 + a\Delta m_3 + 2\Delta s)e^{\Delta s} \end{aligned} \tag{14}$$

for some constants $h_1, \cdots, h_6$ and a, which satisfy one of the following three conditions:

(1) $h_5 = h_6 = 0$.

(2) $h_6 = 0$ and $f^2(\Delta s) + af^3(\Delta s) = 2(e^{\Delta s} - \Delta s - 1)$ for the constant a.

(3) $h_5 = 0$ and $f^2(\Delta s) + af^3(\Delta s) = 2(\Delta s e^{\Delta s} - e^{\Delta s} + 1)$ for the constant a.

The proof is provided in Appendix 1.

Proposition 3.3 shows that three terms are satisfying the AP and containing m_3; the 4th, 5th and 6th terms in Equation (14). The AP of the 4th term Δm_3 is trivial because it is a (non-transformed) given process. Except for the 4th term, Δm_3 always appears with Δm_2 and a as $\Delta m_2 + a\Delta m_3$ with specific forms of $f^2(\Delta s) + af^3(\Delta s)$, which satisfies the condition of a generalized second moment. Proposition 3.3 is therefore equivalent to a result under information set $x = (s, \tilde{m}_2)$ with a generalized second moment $\tilde{m}_2 = m_2 + am_3$ that is obtained from $\tilde{f}^2 = f^2 + af^3$. It implies that the additional information of m_3 to the information set does not produce any non-trivial function satisfying the AP. Related to this, Corollary 3.4 indicates that there is no realized fourth moment.

Corollary 3.4. When the information set is given by $x = (s, m_2, m_3)$, there is no realized 4-moment.

The proof is similar to that for Corollary 3.4.

4. Arithmetic realized joint cumulants

According to section 3, there is some skepticism about the geometric realized higher-order comoments. However, as mentioned in section 1, financial studies state the importance of the higher-order comoments even above the fourth-order. Different from geometric comoments, arithmetic ones up to the fourth-order are available (recall Table 1). This section provides an investigation of the arithmetic comoments of general orders. Strictly speaking, our goal is to present realized joint cumulants. Because these are lesser-known, let us see their definitions.

Definition 4.1. Cumulants and joint cumulants

The lth cumulant of a random variable Y is defined by

$$\kappa_l(Y) = \frac{\partial^l}{\partial u^l} lnE[\exp(uY)]\bigg|_{u=0}. \tag{15}$$

The joint cumulant of random variables $Y_1, Y_2, \ldots, Y_l$ is defined by

$$\kappa(Y_1, Y_2, \ldots, Y_l) = \frac{\partial^l}{\partial u_1 \partial u_2 \cdots \partial u_l} lnE\left[\exp\left(\sum_{i=1}^{l} u_i Y_i\right)\right]\Bigg|_{u_1=\cdots=u_l=0}. \tag{16}$$

Recent studies such as Khademalomoom *et al.* (2019), Ahmed and Al Mafrachi (2021) and Cui *et al.* (2022) deal with the first six moments. Accordingly, the first six cumulants $\kappa_l(Y)$ are described in the second column of Table 2. The cumulants are kinds of normalized moments because $\kappa_l(Y) = 0$ for $l \geq 3$ when Y follows a normal distribution. Moreover, a cumulant is a joint cumulant of an identical random variable with itself. In other words,

$$\kappa_l(Y) = \kappa(Y_1, \ldots, Y_l), \quad \text{for} \;\; Y_1 = \cdots = Y_l = Y. \tag{17}$$

Moreover, for any constant number a, we have

$$\kappa_l(Y_M + aY) = \sum_{k=0}^{l} a^k \binom{l}{k} \kappa_{l-k,k}(Y_M, Y), \tag{18}$$

where $\kappa_{l-k,k}(Y_M, Y) = \kappa(Y_M, \cdots, Y_M, Y, \cdots, Y)$ with $l-k$ Y_Ms and k Ys, and $\kappa_{l-k,k}(Y_M, Y)$ is linked to the comoment $E[Y_M^{l-k} Y^k]$. For example, $\kappa_{0,1}(Y_M, Y)$, ... and $\kappa_{5,1}(Y_M, Y)$ are described in the third column of Table 2.

Fukasawa and Matsushita (2021) present the relationships between cumulants and the AP, and the result is summarized as follows.

$$\mathrm{E}_0\left[\sum_{j=1}^{N} B_L\left(X_{t_j} - X_{t_{j-1}}\right)\right] = E_0[B_L(X_T - X_0)] = {}^0\kappa_L(S_T), [5] \tag{19}$$

where B_L is the Lth complete Bell polynomial defined as

$$B_L(y_1, \cdots, y_L) = \frac{\partial^L}{\partial u^L} \exp\left(\sum_{l=1}^{L} \frac{u^l}{l!} y_l\right)\Bigg|_{u=0}, \tag{20}$$

and $X_t = (S_t, M_t^{(2)}, M_t^{(3)}, \ldots, M_t^{(L-1)}, 0)$ with $M_t^{(l)} = {}^t\kappa_l(S_T)$. Equation (19) implies that

$$\sum_{j=1}^{N} B_L\left(X_{t_j} - X_{t_{j-1}}\right) \tag{21}$$

is an unbiased estimator of ${}^0\kappa_L(S_T)$. Therefore, it can overcome drawbacks of conventional realized moments. Accordingly, the authors name Equation (21) the realized Lth cumulant. For illustration, the realized cumulants of orders 2–6 are presented in Table 3. As stated in Neuberger (2012), Amaya *et al.* (2015), and Bae and Lee (2021), when l is not two, each summand requires additional terms more than $(\Delta S_{t_j})^l$. For example, $\Delta M_{t_j}^{(2)} \Delta S_{t_j}$ can reflect leverage effect when $l = 3$, and $\Delta M_{t_j}^{(2)} (\Delta S_{t_j})^2$ can reflect volatility structure when $l = 4$.

By extending Fukasawa and Matsushita (2021), we provide realized joint cumulants in Proposition 4.2.

Proposition 4.2. For martingale processes $S_{1,t}$ and $S_{2,t}$, let us define $cM_{L-1,1}^{real}(S_1, S_2)$ as follows

$$cM_{L-1,1}^{real}(S_1, S_2) = \sum_{j=1}^{N} \sum_{k=1}^{L-1} \binom{L-1}{k} B_{L-k}\left(\Delta M_{t_j}^{(1,0)}, \cdots, \Delta M_{t_j}^{(L-k,0)}\right) \Delta M_{t_j}^{(k-1,1)} \tag{22}$$

Table 2.
The first six cumulants

l	$\kappa_l(Y)$	$\kappa_{l-1,1}(Y_M, Y)$
1	$E[Y]$	$E[Y]$
2	$E[\widehat{Y}^2]$	$E[\widehat{Y}_M \widehat{Y}]$
3	$E[\widehat{Y}^3]$	$E[\widehat{Y}_M^2 \widehat{Y}]$
4	$E[\widehat{Y}^4] - 3E[\widehat{Y}^2]^2$	$E[\widehat{Y}_M^3 \widehat{Y}] - 3E[\widehat{Y}_M^2]E[\widehat{Y}_M \widehat{Y}]$
5	$E[\widehat{Y}^5] - 10E[\widehat{Y}^3]E[\widehat{Y}^2]$	$E[\widehat{Y}_M^4 \widehat{Y}] - 6E[\widehat{Y}_M^2]E[\widehat{Y}_M^2 \widehat{Y}] - 4E[\widehat{Y}_M^3]E[\widehat{Y}_M \widehat{Y}]$
6	$E[\widehat{Y}^6] - 15E[\widehat{Y}^4]E[\widehat{Y}^2] - 10E[\widehat{Y}^3]^2 + 30E[\widehat{Y}^2]^3$	$E[\widehat{Y}_M^5 \widehat{Y}] - 10E[\widehat{Y}_M^3 \widehat{Y}]E[\widehat{Y}_M^2] - 5E[\widehat{Y}_M^4]E[\widehat{Y}_M \widehat{Y}] - 10E[\widehat{Y}_M^3]E[\widehat{Y}_M^2 \widehat{Y}] + 30E[\widehat{Y}_M^2]^2 E[\widehat{Y}_M \widehat{Y}]$

Note(s): The second column presents cumulants of a random variable Y. The third column presents joint cumulants $\kappa_{l-1,1}(Y_M, Y)$ for $l = 1,2,\ldots,6$. $\widehat{Y}$ and $\widehat{Y}_M$ in the row 3–7 are $Y - E[Y]$ and $Y_M - E[Y_M]$, respectively

Table 3. Examples of realized cumulants

l	Realized lth cumulant
2	$\sum_{j=1}^{N}(\Delta S_{t_j})^2$
3	$\sum_{j=1}^{N}((\Delta S_{t_j})^3 + 3\Delta M_{t_j}^{(2)}\Delta S_{t_j})$
4	$\sum_{j=1}^{N}((\Delta S_{t_j})^4 + 6\Delta M_{t_j}^{(2)}(\Delta S_{t_j})^2 + 3(\Delta M_{t_j}^{(2)})^2 + 4\Delta M_{t_j}^{(3)}\Delta S_{t_j})$
5	$\sum_{j=1}^{N}((\Delta S_{t_j})^5 + 10\Delta M_{t_j}^{(2)}(\Delta S_{t_j})^3 + 15(\Delta M_{t_j}^{(2)})^2\Delta S_{t_j} + 10\Delta M_{t_j}^{(3)}(\Delta S_{t_j})^2 + 10\Delta M_{t_j}^{(3)}\Delta M_{t_j}^{(2)} + 5\Delta M_{t_j}^{(4)}\Delta S_{t_j})$
6	$\sum_{j=1}^{N}((\Delta S_{t_j})^6 + 15\Delta M_{t_j}^{(2)}(\Delta S_{t_j})^4 + 20\Delta M_{t_j}^{(3)}(\Delta S_{t_j})^3 + 45(\Delta M_{t_j}^{(2)})^2(\Delta S_{t_j})^2 + 15(\Delta M_{t_j}^{(2)})^3 + 60\Delta M_{t_j}^{(3)}\Delta M_{t_j}^{(2)}\Delta S_{t_j} + 15\Delta M_{t_j}^{(4)}(\Delta S_{t_j})^2 + 10(\Delta M_{t_j}^{(3)})^2 + 15\Delta M_{t_j}^{(4)}\Delta M_{t_j}^{(2)} + 6\Delta M_{t_j}^{(5)}\Delta S_{t_j})$

Note(s): This table describes the realized lth cumulants in Bae and Lee (2021) and Fukasawa and Matsushita (2021). $\Delta M_{t_j}^{(l)}$ in the row 2–6 are $M_{t_j}^{(l)} - M_{t_{j-1}}^{(l)}$. Recall that $S_t = M_t^{(1)}$

Table 4.
The 2–6 realized joint cumulants

l	$cM^{real}_{l-1,1}(S_1, S_2)$
2	$\sum_{j=1}^{N} \Delta S_{1,t_j} \Delta S_{2,t_j}$
3	$\sum_{j=1}^{N} (((\Delta S_{1,t_j})^2 + \Delta M_{t_j}^{(2,0)}) \Delta S_{2,t_j} + 2\Delta S_{1,t_j} \Delta M_{t_j}^{(1,1)})$
4	$\sum_{j=1}^{N} (((\Delta S_{1,t_j})^3 + 3\Delta M_{t_j}^{(2,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(3,0)}) \Delta S_{2,t_j} + 3((\Delta S_{1,t_j})^2 + \Delta M_{t_j}^{(2,0)}) \Delta M_{t_j}^{(1,1)} + 3\Delta S_{1,t_j} \Delta M_{t_j}^{(2,1)})$
5	$\sum_{j=1}^{N} (((\Delta S_{1,t_j})^4 + 6\Delta M_{t_j}^{(2,0)} (\Delta S_{1,t_j})^2 + 3(\Delta M_{t_j}^{(2,0)})^2 + 4\Delta M_{t_j}^{(3,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(4,0)}) \Delta S_{2,t_j} + 4((\Delta S_{1,t_j})^3 + 3\Delta M_{t_j}^{(2,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(3,0)}) \Delta M_{t_j}^{(1,1)} + 6((\Delta S_{1,t_j})^2 + \Delta M_{t_j}^{(2,0)}) \Delta M_{t_j}^{(2,1)} + 4\Delta S_{1,t_j} \Delta M_{t_j}^{(3,1)})$
6	$\sum_{j=1}^{N} (((\Delta S_{1,t_j})^5 + 10\Delta M_{t_j}^{(2,0)} (\Delta S_{1,t_j})^3 + 10\Delta M_{t_j}^{(3,0)} (\Delta S_{1,t_j})^2 + 15(\Delta M_{t_j}^{(2,0)})^2 \Delta S_{1,t_j} + 10\Delta M_{t_j}^{(3,0)} \Delta M_{t_j}^{(2,0)} + 5\Delta M_{t_j}^{(4,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(5,0)}) \Delta S_{2,t_j} + 5((\Delta S_{1,t_j})^4 + 6\Delta M_{t_j}^{(2,0)} (\Delta S_{1,t_j})^2 + 3(\Delta M_{t_j}^{(2,0)})^2$ $+4\Delta M_{t_j}^{(3,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(4,0)}) \Delta M_{t_j}^{(1,1)} + 10((\Delta S_{1,t_j})^3 + 3\Delta M_{t_j}^{(2,0)} \Delta S_{1,t_j} + \Delta M_{t_j}^{(3,0)}) \Delta M_{t_j}^{(2,1)} + 10((\Delta S_{1,t_j})^2 + \Delta M_{t_j}^{(2,0)}) \Delta M_{t_j}^{(3,1)} + 5\Delta S_{1,t_j} \Delta M_{t_j}^{(4,1)})$

Note(s): This table describes detailed forms of the realized lth joint cumulants $cM^{real}_{l-1,1}(S_1, S_2)$ up to order six. $\Delta M_{t_j}^{(l-k,k)}$ in the row 2–6 are $M_{t_j}^{(l-k,k)} - M_{t_{j-1}}^{(l-k,k)}$. Recall that $S_{1,t} = M_t^{(1,0)}$ and $S_{2,t} = M_t^{(0,1)}$

with $\Delta M_{t_j}^{(l-k,k)} = M_{t_j}^{(l-k,k)} - M_{t_{j-1}}^{(l-k,k)}$ and $M_t^{(l-k,k)} = {}^t\kappa_{l-k,k}(S_{1,T}, S_{2,T})$. Then, we have

$$E_0\left[cM_{L-1,1}^{real}(S_1, S_2)\right] = {}^0\kappa_{L-1,1}(S_{1,T}, S_{2,T}). \tag{23}$$

Proof is provided in Appendix 2.

Based on Proposition 4.2, we can measure the relationship between S_1 and S_2 well through $cM_{L-1,1}^{real}(S_1, S_2)$. Because of Equation (23), it is an unbiased estimator of ${}^0\kappa_{L-1,1}(S_{1,T}, S_{2,T})$. Therefore, it can overcome drawbacks of conventional measures. For illustration, detailed forms of $cM_{l-1,1}^{real}(S_1, S_2)$ up to order six are presented in Table 4. Like the result of Table 3, it shows that the fifth joint cumulant $cM_{4,1}^{real}(S_1, S_2)$ requires more than $\sum_{j=1}^{N}(\Delta S_{1,t_j})^4 \Delta S_{2,t_j}$. For example, it additionally requires $\sum_{j=1}^{N} \Delta M_{t_j}^{(4,0)} \Delta S_{2,t_j}$, which is related to covariation between the second asset return and the kurtosis of the first asset return. Similarly, $cM_{5,1}^{real}(S_1, S_2)$ requires more than $\sum_{j=1}^{N}(\Delta S_{1,t_j})^5 \Delta S_{2,t_j}$.

5. Concluding remarks

Neuberger (2012), Bae and Lee (2021), and Fukasawa and Matsushita (2021) demonstrate that realized third geometric moments and realized arithmetic moments of any orders are obtained by combining their lower-order implied moments and comoments. Extending the information set is therefore a natural trial to yield the higher order moments and comoments. Unlike previous studies, we show that geometric lower-order implied comoments do not yield geometric realized fourth moment and third comoment but yield geometric realized covariance only. The main reason for the non-existence is that the extension of the geometric information set does not produce additional non-trivial terms; the productions are only transformations of Neuberger (2012). Although this approach does not yield a meaningful measure, presenting this result can prevent the same trial and error for other scholars.

Furthermore, we yield the arithmetic realized lth joint cumulants, which are linked to $E[(S_{1,T} - S_{1,0})^{l-1}(S_{2,T} - S_{2,0})]$. Several financial theories apply them; for example, the extended CAPM includes $E[(r_M - E[r_M])^{l-1}(r_i - E[r_i])]$ for $l \geq 2$. Given the drawbacks of conventional realized comoments, we believe that empirical studies can use our measure in the future.

Depending on combinations of assets, there are other joint cumulants such as $E[(r_M - E[r_M])^{l-2}(r_i - E[r_i])^2]$ or $E[(r_M - E[r_M])^{l-3}(r_i - E[r_i])^3]$. We do not investigate them because they currently seem irrelevant to financial studies. However, we may obtain them as proof of Proposition 4.2 when the financial studies require them.

Notes

1. They use the realized moments for various purposes. Amaya *et al.* (2015) and Sim (2016) show that realized third moments can explain stock returns. Kim (2016) investigates the forecasting power of implied moments about realized moments. Mei *et al.* (2017) show realized third and fourth moments are related to future volatility. Kinateder and Papavassiliou (2019) show that realized fourth moment can predict sovereign bond returns during a crisis. Ahmed and Al Mafrachi (2021) show that realized moments up to the fifth-order can explain cryptocurrency returns.
2. Implied moments can be obtained from options (Bakshi and Madan, 2000; Bakshi *et al.*, 2003; Kang *et al.*, 2009; Neuberger, 2012).
3. Cumulants are normalized moments. See section 4 for details.
4. The rest of this section is preliminary of section 3 that proves the non-existence of the geometric realized comoments. Therefore, readers that are only interested in the form of the realized comoments should move to section 4.

5. A left superscript t of κ means a time-t conditional one. For example, ${}^{t}\kappa_L(Y) = \frac{\partial^L}{\partial u^L} lnE_t[\exp(uY)]\Big|_{u=0}$.
6. The ten coefficients, $b_0, \cdots, b_5, b_{12}, b_{14}, b_{23}, b_{25}$, and b_{26} are replaced with $d_5, \cdots, d_{14}$. More precisely, $(d_5, d_8, d_{12}, d_{13})$ replace $(b_1, b_4, b_{12}, b_{25})$, $(d_6, d_9, d_{11}, d_{14})$ replace $(b_2, b_3, b_{14}, b_{26})$, d_{10} replaces b_{23}, and d_7 replaces b_0, given b_3 and b_{12}.

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Appendix 1

Proofs for Propositions and Corollaries in Section 3

Beginnings of the proofs for Propositions 3.1 and 3.3 are identical. We denote them as common property A as follows:

Common property A: A common necessary condition of g that satisfies the aggregation property.

Consider a vector-valued process $\{(\ln S_1(t), \ln S_2(t), M(t)) : t = 0, 1, 2\}$. In addition, let

$$\begin{array}{lllllll} (\ln S_1, \ln S_2, M): & (0,0,m) & \rightarrow & \left\{\begin{array}{l} (s_{1,1}, s_{2,1}, \alpha) \\ \left(s_{1,2}, s_{2,2}, \vec{0}\right) \\ \vdots \\ \left(s_{1,n}, s_{2,n}, \vec{0}\right) \end{array}\right. & \begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \rightarrow \end{array} & \begin{array}{l} \left(s_{1,1}+\eta_1, s_{2,1}+\eta_2, \vec{0}\right) \\ \left(s_{1,2}, s_{2,2}, \vec{0}\right) \\ \vdots \\ \left(s_{1,n}, s_{2,n}, \vec{0}\right) \end{array} & \begin{array}{l} \Pr = \pi_1 \\ \Pr = \pi_2 \\ \vdots \\ \Pr = \pi_n \end{array} \\ t: & 0 & \rightarrow & 1 & \rightarrow & 2 & \end{array} \tag{A1}$$

with $\sum_{j=1}^{n} \pi_j = 1$, $\sum_{j=1}^{n} \pi_j \exp(s_{i,j}) = 1$, $\mathrm{E}[\exp(\eta_i)] = 1$, $\mathrm{E}[f^{k,l}(\eta_1, \eta_2)] = \alpha_{k,l}$, $\alpha = (\alpha_{2,0}, \alpha_{1,1}, \alpha_{0,2}, \alpha_{0,3})$ and $m = (m_{2,0}, m_{1,1}, m_{0,2}, m_{0,3})$ where

$$m_{k,l} = \pi_1 \mathrm{E}\left[f^{k,l}(s_{1,1}+\eta_1, s_{2,1}+\eta_2)\right] + \sum_{j=2}^{n} \pi_j f^{k,l}(s_{1,j}, s_{2,j}), \tag{A2}$$

and $f^{k,l}$ is a generalized moment function such that $f^{k,l}(0,0) = 0$, $\lim_{(a,b)\to(0,0)} \frac{f^{k,l}(a,b)}{a^k b^l} = 1$, $f^{k,l}(a,b) = f^{l,k}(b,a)$, and $f^k(a) = f^{k,0}(a,b)$.

When the process satisfies the aggregation property, we have

$$E[g(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -m)] = g(s_{1,1}, s_{2,1}, \alpha - m) + E[g(\eta_1, \eta_2, -\alpha)] \tag{A3}$$

with $g(0, \cdots, 0) = 0$. Differentiating Equation (A3) with respect to the $(k-2)$th term of m, we obtain

$$E[g_k(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -m)] = g_k(s_{1,1}, s_{2,1}, \alpha - m), \quad \text{for} \quad k = 3, \cdots, 6, \tag{A4}$$

where g_k is a partial differentiation with respect to the kth term. By substituting $(s_{1,1}, s_{2,1}) = (0,0)$ and $m = \alpha$ into Equation (A4), we obtain:

$$E[g_k(\eta_1, \eta_2, -\alpha)] = g_k\left(0, 0, \vec{0}\right), \quad \text{for} \quad k = 3, \cdots, 6. \tag{A5}$$

Then, by Lagrangian, we have

$$\begin{aligned} g_k(s_1, s_2, M) = {} & a_{k,0} + A_{k,1}(M)(e^{s_1} - 1) + A_{k,2}(M)(e^{s_2} - 1) + A_{k,3}(M)\left(f^2(s_1) + M_{2,0}\right) \\ & + A_{k,4}(M)\left(f^{1,1}(s_1, s_2) + M_{1,1}\right) + A_{k,5}(M)\left(f^2(s_2) + M_{0,2}\right) \\ & + A_{k,6}(M)\left(f^3(s_1) + M_{3,0}\right) \end{aligned} \tag{A6}$$

where $a_{k,0}$ is a constant and $A_{k,1}, \cdots, A_{k,6}$ are functions of M. If we substitute Equation (A6), $(s_{1,1}, s_{2,1}) = (0,0)$ and $m = \alpha$ except for the $(l-2)$th term into Equation (A4), we obtain:

$$A_{k,l}(-m)(\alpha_{l-2}-m_{l-2}) = A_{k,l}(0,\cdots,0,\alpha_{l-2}-m_{l-2},0,\cdots,0)(\alpha_{l-2}-m_{l-2}). \tag{A7}$$

Because π, $s_{i,j}$ and α are arbitrary, $A_{k,3}(M),\cdots,A_{k,6}(M)$ are constants. Thus, Equation (A7) is can be rewritten with notations of $a_{k,3},\ldots,a_{k,6}$, as follows:

$$g_k(s_1,s_2,M) = a_{k,0} + A_{k,1}(M)(e^{s_1}-1) + A_{k,2}(M)(e^{s_2}-1) + a_{k,3}\left(f^2(s_1)+M_{2,0}\right) + a_{k,4}\left(f^{1,1}(s_1,s_2)+M_{1,1}\right) + a_{k,5}\left(f^2(s_2)+M_{0,2}\right) + a_{k,6}\left(f^3(s_1)+M_{3,0}\right). \tag{A8}$$

To investigate $A_{k,1}(M)$ and $A_{k,2}(M)$, let us substitute (A8) into (A4) and differentiate it with respect to m_l. It yields

$$\frac{\partial A_{k,1}(-m)}{\partial m_l} = \frac{\partial A_{k,1}(\alpha-m)}{\partial m_l}, \frac{\partial A_{k,2}(-m)}{\partial m_l} = \frac{\partial A_{k,2}(\alpha-m)}{\partial m_l} \quad \text{for } l=3,\ldots,6. \tag{A9}$$

Therefore, $A_{k,1}(M)$ and $A_{k,2}(M)$ are affine functions. Accordingly, (A8) is represented as follows:

$$\begin{aligned} g_k(s_1,s_2,M) = {} & a_{k,0} + (b_{k,0}+b_{k,1}M_{2,0}+b_{k,2}M_{1,1}+b_{k,3}M_{0,2}+b_{k,4}M_{3,0})(e^{s_1}-1) \\ & + (c_{k,0}+c_{k,1}M_{2,0}+c_{k,2}M_{1,1}+c_{k,3}M_{0,2}+c_{k,4}M_{3,0})(e^{s_2}-1) \\ & + a_{k,3}\left(f^2(s_1)+M_{2,0}\right) + a_{k,4}\left(f^{1,1}(s_1,s_2)+M_{1,1}\right) + a_{k,5}\left(f^2(s_2)+M_{0,2}\right) \\ & + a_{k,6}\left(f^3(s_1)+M_{3,0}\right). \end{aligned} \tag{A10}$$

Proof for Proposition 3.1

(Proof for the first statement)

We use the common property A with restricting the $M=(V_1,V_2,V_c)$ with $V_1=M_{2,0}$, $V_2=M_{0,2}$ and $V_c=M_{1,1}$. Similarly, we use notations $m=(v_1,v_2,v_c)$ and $\alpha=(\alpha_1,\alpha_2,\alpha_c)$. In addition, f and f_c replace f^2 and $f^{1,1}$, respectively. By integrating (A10) with respect to V_1, V_c and V_2, we can obtain three different forms of $g(s_1,s_2,V_1,V_2,V_c)$ as follows.

$$\begin{aligned} g(s_1,s_2,V_1,V_2,V_c) = {} & a_{1,0}V_1 + \left(b_{1,0}V_1 + \frac{1}{2}b_{1,1}V_1^2 + b_{1,2}V_1V_c + b_{1,3}V_1V_2\right)(e^{s_1}-1) \\ & + \left(c_{1,0}V_1 + \frac{1}{2}c_{1,1}V_1^2 + c_{1,2}V_1V_c + c_{1,3}V_1V_2\right)(e^{s_2}-1) \\ & + a_{1,3}\left(f(s_1)V_1 + \frac{1}{2}V_1^2\right) + a_{1,4}(f_c(s_1,s_2)V_1 + V_1V_c) \\ & + a_{1,5}(f(s_2)V_1 + V_1V_2) + g^1(s_1,s_2,V_2,V_c). \end{aligned} \tag{A11}$$

$$\begin{aligned} g(s_1,s_2,V_1,V_2,V_c) = {} & a_{2,0}V_c + \left(b_{2,0}V_c + b_{2,1}V_1V_c + \frac{1}{2}b_{2,2}V_c^2 + b_{2,3}V_cV_2\right)(e^{s_1}-1) \\ & + \left(c_{2,0}V_c + c_{2,1}V_1V_c + \frac{1}{2}c_{2,2}V_c^2 + c_{2,3}V_cV_2\right)(e^{s_2}-1) \\ & + a_{2,3}(f(s_1)V_c + V_1V_c) + a_{2,4}\left(f_c(s_1,s_2)V_c + \frac{1}{2}V_c^2\right) \\ & + a_{2,5}(f(s_2)V_c + V_cV_2) + g^2(s_1,s_2,V_1,V_c). \end{aligned} \tag{A12}$$

$$g(s_1,s_2,V_1,V_2,V_c) = a_{3,0}V_2 + \left(b_{3,0}V_2 + b_{3,1}V_1V_2 + b_{3,2}V_cV_2 + \frac{1}{2}b_{3,3}V_2^2\right)(e^{s_1}-1) + \left(c_{3,0}V_2 + c_{3,1}V_1V_2 + c_{3,2}V_cV_2 + \frac{1}{2}c_{3,3}V_2^2\right)(e^{s_2}-1) + a_{3,3}(f(s_1)V_2 + V_1V_2) + a_{3,4}(f_c(s_1,s_2)V_2 + V_cV_2) + a_{3,5}\left(f(s_2)V_2 + \frac{1}{2}V_2^2\right) + g^3(s_1,s_2,V_1,V_2). \tag{A13}$$

with some functions g^1, g^2, and g^3. By combining Equations (A11), (A12) and (A13), we obtain

$$\begin{aligned} g(s_1,s_2,V_1,V_2,V_c) = {} & b_0V_c + b_1V_1 + b_2V_2 + (e^{s_1}-1)(b_3V_c + b_4V_1 + b_5V_2 \\ & + b_6V_cV_1 + b_7V_cV_2 + b_8V_1V_2 + b_9V_c^2 + b_{10}V_1^2 + b_{11}V_2^2) \\ & + (e^{s_2}-1)(b_{12}V_c + b_{13}V_1 + b_{14}V_2 + b_{15}V_cV_1 \\ & + b_{16}V_cV_2 + b_{17}V_1V_2 + b_{18}V_c^2 + b_{19}V_1^2 + b_{20}V_2^2) \\ & + b_{21}(f(s_1)V_c + V_1V_c + V_1f_c(s_1,s_2)) \\ & + b_{22}(f(s_2)V_c + V_2V_c + V_2f_c(s_1,s_2)) \\ & + b_{23}(f(s_2)V_1 + V_1V_2 + f(s_1)V_2) + b_{24}(2f_c(s_1,s_2) + V_c)V_c \\ & + b_{25}(2f(s_1) + V_1)V_1 + b_{26}(2f(s_2) + V_2)V_2 + g^s(s_1,s_2) \end{aligned} \tag{A14}$$

for some constants $b_1,\cdots,b_{26}$ and a function g^s such that $g^s(0,0)=0$.

Based on (η_1,η_2), which are in Equation (A1), let us construct $(\eta_1^p,\eta_2^p) = \begin{cases} (\eta_1,\eta_2) & \Pr = p \\ (0,0), & \Pr = 1-p \end{cases}$ for a constant p in [0,1]. For $i\in\{1,2\}$, we have $E[e^{\eta_i^p}]=1$, $E[f(\eta_i^p)] = \alpha_i p$ and $E[f_c(\eta_1^p,\eta_2^p)] = \alpha_c p$. Then, by substituting Equation (A14) into (A3) and (η_1^p,η_2^p) into (η_1,η_2), we obtain:

$$\begin{aligned} 0 = {} & p(e^{s_{11}}-1)\begin{pmatrix} b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2 + b_6(\alpha_c\alpha_1p - \alpha_1v_c - \alpha_cv_1) \\ + b_7(\alpha_2\alpha_cp - \alpha_2v_c - \alpha_cv_2) + b_8(\alpha_1\alpha_2p - \alpha_2v_1 - \alpha_1v_2) \\ + b_9(\alpha_c^2p - 2\alpha_cv_c) + b_{10}(\alpha_1^2p - 2\alpha_1v_1) + b_{11}(\alpha_2^2p - 2\alpha_2v_2) \end{pmatrix} \\ & + p(e^{s_{21}}-1)\begin{pmatrix} b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2 + b_{15}(\alpha_c\alpha_1p - \alpha_1v_c - \alpha_cv_1) \\ + b_{16}(\alpha_2\alpha_cp - \alpha_2v_c - \alpha_cv_2) + b_{17}(\alpha_1\alpha_2p - \alpha_2v_1 - \alpha_1v_2) \\ + b_{18}(\alpha_c^2p - 2\alpha_cv_c) + b_{19}(\alpha_1^2p - 2\alpha_1v_1) + b_{20}(\alpha_2^2p - 2\alpha_2v_2) \end{pmatrix} \\ & + pb_{21}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1f_c(s_{11},s_{21}) \\ + (E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_1 \end{pmatrix} \\ & + pb_{22}\begin{pmatrix} (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2f_c(s_{11},s_{21}) \\ + (E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_2 \end{pmatrix} \\ & + pb_{23}\begin{pmatrix} (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_1 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2 \\ + (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_2 \end{pmatrix} \\ & + 2pb_{24}((E[f_c(s_{11}+\eta_1,s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_c + f_c(s_{11},s_{21})\alpha_c) \\ & + 2pb_{25}((E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1) \\ & + 2pb_{26}((E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2) \\ & - pE[g^s(s_{11}+\eta_1,s_{21}+\eta_2)] + pg^s(s_1,s_2) + pE[g^s(\eta_1,\eta_2)] \end{aligned} \tag{A15}$$

Because (A15) holds for arbitrary p, the coefficient of p^2 should be zero.

$$\begin{aligned}0 &= (e^{s_{11}}-1)\left(b_6\alpha_c\alpha_1 + b_7\alpha_2\alpha_c + b_8\alpha_1\alpha_2 + b_9\alpha_c^2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2\right)\\ &\quad + (e^{s_{21}}-1)\left(b_{15}\alpha_c\alpha_1 + b_{16}\alpha_2\alpha_c + b_{17}\alpha_1\alpha_2 + b_{18}\alpha_c^2 + b_{19}\alpha_1^2 + b_{20}\alpha_2^2\right)\end{aligned} \tag{A16}$$

Furthermore, because s_{11} and s_{21} are arbitrary, we have:

$$b_6\alpha_c\alpha_1 + b_7\alpha_2\alpha_c + b_8\alpha_1\alpha_2 + b_9\alpha_c^2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2 = 0 \tag{A17}$$

$$b_{15}\alpha_c\alpha_1 + b_{16}\alpha_2\alpha_c + b_{17}\alpha_1\alpha_2 + b_{18}\alpha_c^2 + b_{19}\alpha_1^2 + b_{20}\alpha_2^2 = 0 \tag{A18}$$

Given α_1 and α_2, we can construct arbitrary α_c. Therefore, Equation (A17) yields:

$$b_9 = b_6\alpha_1 + b_7\alpha_2 = b_8\alpha_1\alpha_2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2 = 0 \tag{A19}$$

Adopting this logic to Equation (A18) instead of (A17) and to α_1 or α_2 instead of α_c, we can obtain

$$b_6 = b_7 = \cdots = b_{11} = 0 \text{ and } b_{15} = b_{16} = \cdots = b_{20} = 0 \tag{A20}$$

Additionally, because the coefficient of p in Equation (A15) is zero, we have:

$$\begin{aligned}0 &= (e^{s_{11}}-1)(b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2) + (e^{s_{21}}-1)(b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2)\\ &\quad + b_{21}\begin{pmatrix}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_1\end{pmatrix}\\ &\quad + b_{22}\begin{pmatrix}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2 f_c(s_{11},s_{21})\\ +(E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_2\end{pmatrix}\\ &\quad + b_{23}\begin{pmatrix}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_1 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2\\ +(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_2\end{pmatrix}\\ &\quad + 2b_{24}((E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_c + f_c(s_{11},s_{21})\alpha_c)\\ &\quad + 2b_{25}((E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1)\\ &\quad + 2b_{26}((E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2)\\ &\quad - E[g^s(s_{11}+\eta_1, s_{21}+\eta_2)] + g^s(s_1,s_2) + E[g^s(\eta_1,\eta_2)]\end{aligned} \tag{A21}$$

Because v_c is arbitrary, the coefficient of v_c is zero. Thus, we have

$$\begin{aligned}&b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)\\ &\quad + 2b_{24}(E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c) = 0\end{aligned} \tag{A22}$$

Now, consider a random variable η_3 with $\eta_3 \overset{d}{=} \eta_2$ and $E[f_c(\eta_1,\eta_3)] \neq E[f_c(\eta_1,\eta_2)]$. Then,

$$\begin{aligned}&b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_3)] - f(s_{21}) - \alpha_2)\\ &\quad + 2b_{24}(E[f_c(s_{11}+\eta_1, s_{21}+\eta_3)] - f_c(s_{11},s_{21}) - E[f_c(\eta_1,\eta_3)]) = 0\end{aligned} \tag{A23}$$

By subtracting Equation (A23) from Equation (A22), one can see that $b_{24} = 0$ or

$$E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] = E[f_c(s_{11}+\eta_1, s_{21}+\eta_3)] + E[f_c(\eta_1,\eta_2)] - E[f_c(\eta_1,\eta_3)]. \tag{A24}$$

When we substitute

$$(\eta_1,\eta_2,\eta_3) = \begin{cases}(\ln(1+k), \ln(1+k), \ln(1-k)) & \Pr = 1/2\\ (\ln(1-k), \ln(1-k), \ln(1+k)) & \Pr = 1/2\end{cases} \tag{A25}$$

into Equation (A24) and multiply both sides of the equation with $\frac{1}{2k^2}$, and take the limit with $k \to 0$, we get

$$\frac{\partial^2 f_c(x,y)}{\partial x \partial y} = 1 \quad (A26)$$

Hence,

$$f_c(s_{11}, s_{21}) = s_{11}s_{21} + F_1(s_{11}) + F_2(s_{21}) \quad (A27)$$

for some functions F_1 and F_2. Then, applying the condition of $\lim_{(x,y)\to(0,0)} \frac{f_c(x,y)}{xy} = 1$, we obtain $f_c(s_{11}, s_{21}) = s_{11}s_{21}$. By substituting it into Equation (A22), we obtain the following equation:

$$b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) + 2b_{24}(s_{11}E[\eta_2] + s_{21}E[\eta_1]) = 0 \quad (A28)$$

When $s_{11} = 0$, Equation (A28) becomes $b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) + 2b_{24}s_{21}E[\eta_1] = 0$. Because η_1 can be chosen independently on s_{21} and η_2,

$$b_{24} = 0. \quad (A29)$$

The logic between Equations (A22) and (A29) shows that multiplier of $E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c$ is zero. For alternatives of Equation (A22), as the coefficients of v_1 and v_2 instead of v_c in Equation (A21), the same logic yields

$$b_{21} = b_{22} = 0. \quad (A30)$$

Because the coefficients of v_1 and v_2 in Equation (A21) are 0, equations (A29) and (A30) implies:

$$\begin{aligned} &b_{23}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) + 2b_{25}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) \\ &= b_{23}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + 2b_{26}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) = 0. \end{aligned} \quad (A31)$$

Substituting $s_{11} = 0$ or $s_{21} = 0$ into the (A31) yields:

$$\begin{aligned} &b_{23}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) = b_{25}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) \\ &= b_{23}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) = b_{26}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) = 0 \end{aligned} \quad (A32)$$

Thus,

$$E[f(s+\eta)] - f(s) - E[f(\eta)] = 0 \ \text{ or } \ b_{23} = b_{25} = b_{26} = 0. \quad (A33)$$

Here, according to Neuberger (2012), $E[f(s+\eta)] - f(s) - E[f(\eta)] = 0$ is equivalent to

$$f(x) = 2(e^x - 1 - x). \quad (A34)$$

In sum, combining (A21) with (A29), (A30) and (A33) yields

$$\begin{aligned} 0 = &(e^{s_{11}} - 1)(b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2) + (e^{s_{21}} - 1)(b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2) \\ &+ b_{23}(f(s_{21})\alpha_1 + f(s_{11})\alpha_2) + 2b_{25}f(s_{11})\alpha_1 + 2b_{26}f(s_{21})\alpha_2 \\ &- E[g^s(s_{11}+\eta_1, s_{21}+\eta_2)] + g^s(s_1, s_2) + E[g^s(\eta_1, \eta_2)] \end{aligned} \quad (A35)$$

To Equation (A35), multiplying $2/k^2$, substituting $(\eta_1, \eta_2) = \begin{cases} (\ln(1+k), 0) & \Pr = 1/2 \\ (\ln(1-k), 0) & \Pr = 1/2 \end{cases}$, and taking the limit yields:

$$0 = 2b_4(e^{s_{11}} - 1) + 2b_{13}(e^{s_{21}} - 1) + 4b_{23}(e^{s_{21}} - 1 - s_{21}) + 8b_{25}(e^{s_{11}} - 1 - s_{11}) - \frac{\partial^2 g^s(s_{11}, s_{21})}{\partial s_{11}^2} + \frac{\partial g^s(s_{11}, s_{21})}{\partial s_{11}} + \frac{\partial^2 g^s(0,0)}{\partial s_{11}^2} - \frac{\partial g^s(0,0)}{\partial s_{11}}. \tag{A36}$$

Similarly, when use $(\eta_1, \eta_2) = \begin{cases} (0, \ln(1+k)) & \Pr = 1/2 \\ (0, \ln(1-k)) & \Pr = 1/2 \end{cases}$, we can obtain

$$0 = 2b_5(e^{s_{11}} - 1) + 2b_{14}(e^{s_{21}} - 1) + 4b_{23}(e^{s_{11}} - 1 - s_{11}) + 8b_{26}(e^{s_{21}} - 1 - s_{21}) - \frac{\partial^2 g^s(s_{11}, s_{21})}{\partial s_{21}^2} + \frac{\partial g^s(s_{11}, s_{21})}{\partial s_{21}} + \frac{\partial^2 g^s(0,0)}{\partial s_{21}^2} - \frac{\partial g^s(0,0)}{\partial s_{21}}. \tag{A37}$$

Alternatively, let us multiply $\frac{1}{2k^2}$ to Equation (A35), substitute (A38) and (A39) into Equation (A35), and subtract the equation obtained by the former substitution from that obtained by the latter; then, by taking limits, we get (A40).

$$(\eta_1, \eta_2) = \begin{cases} (\ln(1+k), \ln(1+k)) & \Pr = 1/2 \\ (\ln(1-k), \ln(1-k)) & \Pr = 1/2 \end{cases} \tag{A38}$$

$$(\eta_1, \eta_2) = \begin{cases} (\ln(1+k), \ln(1-k)) & \Pr = 1/2 \\ (\ln(1-k), \ln(1+k)) & \Pr = 1/2 \end{cases} \tag{A39}$$

$$0 = b_3(e^{s_{11}} - 1) + b_{12}(e^{s_{21}} - 1) - \frac{\partial^2 g^s(s_{11}, s_{21})}{\partial s_{11} s_{21}} + \frac{\partial^2 g^s(0,0)}{\partial s_{11} s_{21}} \tag{A40}$$

Then, the solutions of Equation (A40), (A36) and (A37) are given as

$$g^s(x,y) = b_3(e^x - x)y + b_{12}(e^y - y)x + h_1(x) + h_2(y) + b_{27}xy \tag{A41}$$

$$\begin{aligned} g^s(x,y) = {} & 2b_4(e^x x - e^x + x) - 2b_{13}(e^y - 1)x - 4b_{23}x(e^y - y - 1) \\ & + 4b_{25}\left(2e^x x - 2e^x + x^2 + 4x\right) + e^x h_3(y) + h_4(y) + b_{28}x \end{aligned} \tag{A42}$$

$$\begin{aligned} g^s(x,y) = {} & 2b_{14}(e^y y - e^y + y) - 2b_5(e^x - 1)y - 4b_{23}y(e^x - x - 1) \\ & + 4b_{26}\left(2e^y y - 2e^y + y^2 + 4y\right) + e^y h_5(x) + h_6(x) + b_{29}y \end{aligned} \tag{A43}$$

for some functions $h_1, \cdots, h_6$ and constants b_i. Therefore, $g^s(x,y)$ is a linear combination of $e^x y$, $e^y x$, xy, $e^x x$, e^x, x^2, x, $e^y y$, e^y, y^2, y and 1. The consistency in the coefficients of $e^x y$ and $e^y x$ requires $b_5 = -\frac{1}{2}b_3 - 2b_{23}$ and $b_{13} = -\frac{1}{2}b_{12} - 2b_{23}$. Thus, by Equation (A14), (A20), (A29), (A30), (A34), (A41), (A42), (A43) and $g^s(0,0) = 0$, g and g^s are given by

$$\begin{aligned} g(s_1, s_2, V_1, V_2, V_c) = {} & b_0 V_c + b_1 V_1 + b_2 V_2 \\ & + \left(b_3 V_c + b_4 V_1 - \left(\frac{1}{2}b_3 + 2b_{23}\right)V_2\right)(e^{s_1} - 1) \\ & + \left(b_{12} V_c - \left(\frac{1}{2}b_{12} + 2b_{23}\right)V_1 + b_{14} V_2\right)(e^{s_2} - 1) \\ & + b_{23}(2(e^{s_2} - s_2 - 1)V_1 + V_1 V_2 + 2(e^{s_1} - s_1 - 1)V_2) \\ & + b_{25}(4(e^{s_1} - s_1 - 1) + V_1)V_1 + b_{26}(4(e^{s_2} - s_2 - 1) + V_2)V_2 + g^s(s_1, s_2), \end{aligned} \tag{A44}$$

$$\begin{aligned} g^s(s_1, s_2) = {} & d_1(e^{s_1} - 1) + d_2 s_1 + d_3(e^{s_2} - 1) + d_4 s_2 + 4b_{23}s_1 s_2 + 4b_{25}s_1^2 \\ & + 4b_{26}s_2^2 + b_3 e^{s_1} s_2 + b_{12} e^{s_2} s_1 + (2b_4 + 8b_{25})e^{s_1} s_1 + (2b_{14} + 8b_{26})e^{s_2} s_2. \end{aligned} \tag{A45}$$

(A44) and (A45) can be arranged as

$$\begin{aligned} g(s_1, s_2, V_1, V_2, V_c) = {} & d_1(e^{s_1}-1) + d_2 s_1 + d_3(e^{s_2}-1) + d_4 s_2 + d_5 V_1 + d_6 V_2 + d_7 V_c \\ & + d_8(V_1 - 2s_1)^2 + d_9(V_2 - 2s_2)^2 + d_{10}(V_1 - 2s_1)(V_2 - 2s_2) \\ & + d_{11} e^{s_1}(2V_c - V_2 + 2s_2) + d_{12} e^{s_2}(2V_c - V_1 + 2s_1) \\ & + d_{13} e^{s_1}(V_1 + 2s_1) + d_{14} e^{s_2}(V_2 + 2s_2) \end{aligned} \tag{A46}$$

where $d_5 = b_1 - b_4 + \frac{1}{2}b_{12} - 4b_{25}$, $d_6 = b_2 + \frac{1}{2}b_3 - b_{14} - 4b_{26}$, $d_7 = b_0 - b_3 - b_{12}$, $d_8 = b_{25}$, $d_9 = b_{26}$, $d_{10} = b_{23}$, $d_{11} = \frac{1}{2}b_3$, $d_{12} = \frac{1}{2}b_{12}$, $d_{13} = b_4 + 4b_{25}$ and $d_{14} = b_{14} + 4b_{26}$ [6].

Substituting these into equation (A3) yields the following:

$$\begin{aligned} 0 = {} & 2d_8(-v_1 - 2s_{11} + \alpha_1)(\alpha_1 + 2E[\eta_1]) + 2d_9(-v_2 - 2s_{21} + \alpha_2)(\alpha_2 + 2E[\eta_2]) \\ & + d_{10}((-v_1 - 2s_{11} + \alpha_1)(\alpha_2 + 2E[\eta_2]) + (-v_2 - 2s_{21} + \alpha_2)(\alpha_1 + 2E[\eta_1])) \\ & + (e^{s_{11}} - 1)(d_{13}(\alpha_1 - 2E[\eta_1 e^{\eta_1}]) + d_{11}(2\alpha_c - 2E[\eta_2 e^{\eta_1}] - \alpha_2)) \\ & + (e^{s_{21}} - 1)(d_{14}(\alpha_2 - 2E[\eta_2 e^{\eta_2}]) + d_{12}(2\alpha_c - 2E[\eta_1 e^{\eta_2}] - \alpha_1)) \end{aligned} \tag{A47}$$

Since coefficients of v_1 and v_2 are zero, $E[f(\eta)] = E[-2\eta]$ or $d_8 = d_9 = d_{10} = 0$. In addition, because s_{11} and s_{21} are arbitrary,

$$0 = d_{13}(\alpha_1 - 2E[\eta_1 e^{\eta_1}]) + d_{11}(2\alpha_c - 2E[\eta_2 e^{\eta_1}] - \alpha_2) \tag{A48}$$

$$0 = d_{14}(\alpha_2 - 2E[\eta_2 e^{\eta_2}]) + d_{12}(2\alpha_c - 2E[\eta_1 e^{\eta_2}] - \alpha_1) \tag{A49}$$

Conditions of Equations (A47)-(A49) can be fulfilled with one of following five cases.

(1) If d_{11} is not zero, for some constants k_1 and k_2, we have

$$f_c(\eta_1, \eta_2) = \eta_2 e^{\eta_1} + \frac{1}{2}f(\eta_2) + \frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1} - f(\eta_1)) + k_1(e^{\eta_1} - 1) + k_2(e^{\eta_2} - 1). \tag{A50}$$

Then $\frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1} - f(\eta_1)) + k_1(e^{\eta_1} - 1) = 0$ and $\frac{1}{2}f(\eta_2) + k_2(e^{\eta_2} - 1) = -\eta_2$ because $\frac{f_c(x,y)}{xy} \to 1$ as $(x, y) \to (0, 0)$. Accordingly, $k_2 = -1$ because $\frac{f(x)}{x^2} \to 1$ as $x \to 0$. This implies that $k_1 = d_{13} = 0$. Therefore, $f_c(\eta_1, \eta_2) = \eta_2(e^{\eta_1} - 1)$, $f(\eta) = 2(e^{\eta} - \eta - 1)$. Then, $d_{12} = d_{14} = 0$.

(2) Similarly, if d_{12} is not zero, $f_c(\eta_1, \eta_2) = \eta_1(e^{\eta_2} - 1)$, $f(\eta) = 2(e^{\eta} - \eta - 1)$ and $d_{11} = d_{13} = d_{14} = 0$

Alternatively, when $d_{11} = d_{12} = 0$, Equations (A47)-(A49) implies there are three more conditions as follows:

(3) $d_{11} = d_{12} = d_{13} = d_{14} = 0$, $f(\eta) = 2(e^{\eta} - \eta - 1)$, with arbitrary function f_c.

(4) $d_8 = d_9 = d_{10} = d_{11} = d_{12} = 0$, $f(\eta) = 2(\eta e^{\eta} - e^{\eta} + 1)$, with arbitrary function f_c.

(5) $d_8 = d_9 = d_{10} = d_{11} = d_{12} = d_{13} = d_{14} = 0$, with arbitrary functions f and f_c.

(Proof for the second statement)

For the proof of the sufficiency of Equation (6) for the AP, we show that the function g in Equation (6) satisfies the *SAP* (*strong aggregation property*): $E_t[g(X_u - X_r)] = E_t[g(X_u - X_t)] + E_t[g(X_t - X_r)]$, which is stronger condition than the AP of Equation (2). The SAP of the first seven terms in the equation is obvious. The 10th term is a generalization of the 8th and the 9th term and all these three terms do not vanish only if $f(\eta) = 2(e^{\eta} - \eta - 1)$. Thus, SAP of 10th term implies the SAP of 8th and 9th terms. For convenience, let

$$G_{u,t} = \phi_{1,u,t}\phi_{2,u,t} \tag{A51}$$

with

$$\phi_{i,u,t} = (V_{i,u} - V_{i,t} - 2(s_{i,u} - s_{i,t})) \quad \text{for } i \in \{1,2\} \tag{A52}$$

and

$$V_{i,t} \equiv E_t[2(e^{s_{i,T}-s_{i,t}} - (s_{i,T} - s_{i,t}) - 1)] = E_t[-2(s_{i,T} - s_{i,t})] = E_t[V_{i,u} - 2(s_{i,u} - s_{i,t})] \quad \text{for } i \in \{1,2\}. \tag{A53}$$

Then, we have

$$\begin{aligned} E_t[G_{u,0}] &= E_t[(\phi_{1,u,t} + \phi_{1,t,0})(\phi_{2,u,t} + \phi_{2,t,0})] = E_t[\phi_{1,u,t}\phi_{2,u,t} + \phi_{1,t,0}\phi_{2,t,0}] \\ &= E_t[G_{u,t}] + G_{t,0}. \end{aligned} \tag{A54}$$

Thus, we have the SAP of 8th and 9th terms as well as the SAP of 10th term.

Additionally, the SAP of the 11th term under the condition (1) implies that of the 12th term under the condition (2) and the 13th and 14th terms under the condition (4). Thus, we finish this proof by showing Equation (A55).

$$E_t[F_{u,0}] = E_t[F_{u,t}] + F_{t,0}, \quad 0 \le t \le u \le T, \tag{A55}$$

where

$$F_{u,t} = E_t\left[e^{s_{1,u}-s_{1,t}}\left(\widehat{V}_u - \widehat{V}_t + 2(s_{2,u} - s_{2,t})\right)\right] \tag{A56}$$

and

$$\widehat{V}_t \equiv 2E_t[(s_{2,T} - s_{2,t})(e^{s_{1,T}-s_{1,t}} - 1) - (e^{s_{2,T}-s_{2,t}} - (s_{2,T} - s_{2,t}) - 1)]. \tag{A57}$$

Equation (A57) is represented as:

$$\begin{aligned} \widehat{V}_t &= E_t[2e^{s_{1,T}-s_{1,t}}(s_{2,T} - s_{2,t})] \\ &= E_t[2e^{s_{1,T}-s_{1,u}}e^{s_{1,u}-s_{1,t}}(s_{2,T} - s_{2,u})] + E_t[2e^{s_{1,u}-s_{1,t}}(s_{2,u} - s_{2,t})] \\ &= E_t\left[e^{s_{1,u}-s_{1,t}}\left(\widehat{V}_u + 2(s_{2,u} - s_{2,t})\right)\right]. \end{aligned} \tag{A58}$$

It implies

$$E_t[F_{u,t}] = E_t\left[e^{s_{1,u}-s_{1,t}}\left(\widehat{V}_u + 2(s_{2,u} - s_{2,t})\right) - e^{s_{1,u}-s_{1,t}}\widehat{V}_t\right] = 0 \tag{A59}$$

and

$$\begin{aligned} E_t[F_{u,0}] &= E_t\left[e^{s_{1,u}-s_{1,0}}\left(\widehat{V}_u - \widehat{V}_0 + 2(s_{2,u} - s_{2,0})\right)\right] \\ &= E_t\left[e^{s_{1,u}-s_{1,t}}e^{s_{1,t}-s_{1,0}}\left(\widehat{V}_u + 2(s_{2,u} - s_{2,t}) - \widehat{V}_0 + 2(s_{2,t} - s_{2,0})\right)\right] \\ &= E_t\left[e^{s_{1,t}-s_{1,0}}\left(\widehat{V}_t - \widehat{V}_0 + 2(s_{2,t} - s_{2,0})\right)\right] \\ &= F_{t,0}. \end{aligned} \tag{A60}$$

Due to Equations (A59) and (A60), Equation (A55) holds.

■

Proof for Corollary 3.2

If a function is a realized (k,1)-comoment element for $k \in \{1, 2\}$, Equation (11) should be decomposed as

$$g(\Delta s_1, \Delta s_2, \Delta v_1, \Delta v_2, \Delta v_c) = \left(e^{\Delta s_1} - 1\right)\phi_1(\Delta v_1, \Delta v_2, \Delta v_c) + \left(e^{\Delta s_2} - 1\right)\phi_2(\Delta v_1, \Delta v_2, \Delta v_c) + g^r(\Delta s_1, \Delta s_2) \tag{A61}$$

such that $g^r(\Delta s_1, \Delta s_2) = O((\Delta s_1)^k \Delta s_2)$ because of the restrictions $E[e^{\Delta s_1}] = 1$ and $E[e^{\Delta s_2}] = 1$. $(\Delta v_1)^2$ cannot be a part of $(e^{\Delta s_1} - 1)\phi_1(\Delta v_1, \Delta v_2, \Delta v_c)$ or $(e^{\Delta s_2} - 1)\phi_2(\Delta v_1, \Delta v_2, \Delta v_c)$, as well as $g^r(\Delta s_1, \Delta s_2)$, because it is only in $h_8(\Delta v_1 - 2\Delta s_1)^2$; thus, we have $h_8 = 0$. In a similar manner, by considering $(\Delta v_2)^2$ and $\Delta v_1 \Delta v_2$, we have $h_9 = h_{10} = 0$. Accordingly, $e^{\Delta s_1}\Delta s_2$ and $e^{\Delta s_2}\Delta s_1$ are only cross-terms between Δs_1 and Δs_2. Therefore, none of condition (3), (4) or (5) can generate a realized (k,1)-comoment element for $k \in \{1, 2\}$.

Under the condition (1) in Proposition 3.1, $e^{\Delta s_1}\Delta s_2$ is the only cross-term between Δs_1 and Δs_2. In the remaining terms, we have $h_5 = 0$, $h_6 = h_{11}$, and $h_7 = -2h_{11}$ to separate Δv_1, Δv_c, and Δv_2 from $g^r(\Delta s_1, \Delta s_2)$. Then, the remaining term $h_1(e^{\Delta s_1} - 1) + h_2\Delta s_1 + h_3(e^{\Delta s_2} - 1) + h_4\Delta s_2 + 2h_{11}e^{\Delta s_1}\Delta s_2$ is at most $O(\Delta s_1 \Delta s_2)$ as $(\Delta s_1, \Delta s_2) \to (0, 0)$ when $h_1 = h_2 = h_3 = 0$ and $h_4 = -2h_{11}$. It cannot be of $O(s_1^2 s_2)$, and it is a realized (1,1) comoment when $h_4 = -1$.

In a similar manner, under condition (2), the function is at most is at most $O(s_1 s_2)$, and it is a realized comoment when $h_1 = h_3 = h_4 = h_6 = h_8 = h_9 = h_{10} = 0$, $h_2 = h_7 = -1$, $h_5 = h_{12} = 1/2$.

■

Proof for Proposition 3.3

(Proof for the first statement)

We use the common property A by omitting all terms related to the second security. Thus, g is a function of s_1, M_2 and M_3 where $M_2 = M_{2,0}$ and $M_3 = M_{3,0}$. By integrating (A10) with respect to M_2 and M_3, we can obtain two different forms of the function g:

$$\begin{aligned} g(s_1, M_2, M_3) = {} & a_{1,0}M_2 + \left(b_{1,0}M_2 + \frac{1}{2}b_{1,1}M_2^2 + b_{1,4}M_3M_2\right)(e^{s_1} - 1) \\ & + a_{1,3}\left(M_2 f^2(s_1) + \frac{1}{2}M_2^2\right) + a_{1,6}M_2\left(f^3(s_1) + M_3\right) + g^1(s_1, M_3) \end{aligned} \tag{A62}$$

and

$$\begin{aligned} g(s_1, M_2, M_3) = {} & a_{2,0}M_3 + \left(b_{2,0}M_3 + b_{2,1}M_2M_3 + \frac{1}{2}b_{2,4}M_3^2\right)(e^{s_1} - 1) \\ & + a_{2,3}M_3\left(f^2(s_1) + M_2\right) + a_{2,6}\left(f^3(s_1)M_3 + \frac{1}{2}M_3^2\right) + g^2(s_1, M_2) \end{aligned} \tag{A63}$$

with some functions g^1 and g^2. By combining (A62) and (A63), we obtain

$$\begin{aligned} g(s_1, M_2, M_3) = {} & a_1M_2 + a_2M_3 \\ & + \left(a_3M_2 + a_4M_3 + a_5M_2^2 + a_6M_2M_3 + a_7M_3^2\right)(e^{s_1} - 1) \\ & + a_8\left(M_2^2 + 2M_2f^2(s_1)\right) + a_9\left(M_3^2 + 2f^3(s_1)M_3\right) \\ & + 2a_{10}\left(M_2M_3 + f^2(s_1)M_3 + f^3(s_1)M_2\right) + g^s(s_1) \end{aligned} \tag{A64}$$

for some constants $a_1, \cdots, a_{10}$ and a function g^s such that $g^s(0) = 0$. Using the η_1 in Equation (A1), let us construct $\eta_p = \begin{cases} \eta_1 & \Pr = p \\ 0 & \Pr = 1 - p \end{cases}$ for a constant p in [0,1]. Then, by substituting Equation (A64) into (A3) and replacing η with η_p, we obtain:

$$0 = p\begin{pmatrix} (e^{s_{1,1}}-1)\begin{pmatrix} a_3\alpha_2 + a_4\alpha_3 - 2a_5m_2\alpha_2 \\ +a_6(-m_2\alpha_3 - m_3\alpha_2) - 2a_7m_3\alpha_3 \end{pmatrix} \\ + 2a_8\left(\left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right)m_2 + \alpha_2 f^2(s_{1,1})\right) \\ + 2a_9\left(\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right)m_3 + \alpha_3 f^3(s_{1,1})\right) \\ + 2a_{10}\begin{pmatrix} \left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right)m_3 + \alpha_3 f^2(s_{1,1}) \\ +\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right)m_2 + \alpha_2 f^3(s_{1,1}) \end{pmatrix} \\ - E[g^s(s_{1,1}+\eta_1)] + g^s(s_{1,1}) + E[g^s(\eta_1)] \end{pmatrix} + p^2(e^{s_{1,1}}-1)\left(a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2\right) \tag{A65}$$

Because we can set p arbitrary, coefficients of p and p^2 are zero. Therefore, we have

$$a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2 = 0 \tag{A66}$$

and

$$\begin{pmatrix} (e^{s_{1,1}}-1)\begin{pmatrix} a_3\alpha_2 + a_4\alpha_3 - 2a_5m_2\alpha_2 \\ + a_6(-m_2\alpha_3 - m_3\alpha_2) - 2a_7m_3\alpha_3 \end{pmatrix} \\ + 2a_8\left(\left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right)m_2 + \alpha_2 f^2(s_{1,1})\right) \\ + 2a_9\left(\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right)m_3 + \alpha_3 f^3(s_{1,1})\right) \\ + 2a_{10}\begin{pmatrix} \left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right)m_3 + \alpha_3 f^2(s_{1,1}) \\ + \left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right)m_2 + \alpha_2 f^3(s_{1,1}) \end{pmatrix} \\ - E[g^s(s_{1,1}+\eta_1)] + g^s(s_{1,1}) + E[g^s(\eta_1)] \end{pmatrix} = 0. \tag{A67}$$

Equation (A66) implies that

$$a_5 = a_6 = a_7 = 0 \tag{A68}$$

because η_1 is an arbitrary random variable with $E[e^{\eta_1}] = 1$. Also, in Equation (A67), coefficients of m_2 and m_3 are zero because we can set arbitrary values for them. Thus, we have:

$$0 = a_8\left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right) + a_{10}\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right) \tag{A69}$$

$$0 = a_9\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right) + a_{10}\left(f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2\right) \tag{A70}$$

There are three cases that satisfy both (A69) and (A70). We call them condition A.1 as follows:

Condition A.1

(1) $\exists f^3$ such that $\forall(s_1,\eta_1), f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3 = 0$ and $a_8 = a_{10} = 0$,

(2) $\exists(a, f^2, f^3)$ such that $\forall(s_{1,1},\eta_1), f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2 + a(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3) = 0$ with $a_{10} = a_8 a$ and $a_9 = a^2 a_8$,

(3) $a_8 = a_9 = a_{10} = 0$.

First, we check the condition in A.1.(1). Substituting $\eta_1 = \begin{cases} \ln(1+k), & \Pr = 0.5 \\ \ln(1-k), & \Pr = 0.5 \end{cases}$ into $\frac{2}{k^2}\left(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3\right) = 0$, using $\lim_{x\to 0}\frac{f^3(x)}{x^3} = 1$, and taking the limit for $k \to 0$ yield:

$$\left(f^3\right)''(s_{1,1}) - \left(f^3\right)'(s_{1,1}) = 0 \tag{A71}$$

Thus, $f^3(s_{1,1}) = b_1 e^{s_{1,1}} + b_2$ for some constants b_1 and b_2. However, there are no b_1 and b_2 that makes $\lim_{x\to 0}\frac{f^3(x)}{x^3} = 1$. Therefore, condition A.1 (1) is impossible.

Second, let us check condition A.1.(2). Substituting $f^a(x) = f^2(x) + af^3(x)$ and $\eta_1 = \begin{cases} \ln(1+k), & \Pr = 0.5 \\ \ln(1-k), & \Pr = 0.5 \end{cases}$ into $f^2(s_{1,1}+\eta_1) - f^2(s_{1,1}) - \alpha_2 + a(f^3(s_{1,1}+\eta_1) - f^3(s_{1,1}) - \alpha_3) = 0$, using $\lim_{x\to 0}\frac{f^2(x)}{x^2} = 1$ and $\lim_{x\to 0}\frac{f^3(x)}{x^3} = 1$, and taking the limit for $k \to 0$ yield:

$$(f^a)''(s_{1,1}) - (f^a)'(s_{1,1}) - 2 = 0 \tag{A72}$$

Thus, we have $f^a(s_{1,1}) = b_1 e^{s_{1,1}} + b_2 - 2s_{1,1}$ for some constants b_1 and b_2. Because of the conditions $\lim_{x\to 0}\frac{f^2(x)}{x^2} = 1$ and $\lim_{x\to 0}\frac{f^3(x)}{x^3} = 1$, f^2 has the following form:

$$f^2(s) = 2(e^s - s - 1) - af^3(s). \tag{A73}$$

Substituting (A68), (A73), and Condition A.1 (2) into (A67) yields:

$$\begin{aligned} E[g^s(s_{1,1}+\eta_1)] - g^s(s_{1,1}) - E[g^s(\eta_1)] = (e^{s_{1,1}} - 1)(a_3\alpha_2 + a_4\alpha_3) \\ + 4a_8(\alpha_2 + a\alpha_3)(e^{s_{1,1}} - s_{1,1} - 1) \end{aligned} \tag{A74}$$

Equation (A74) with $a_8 = 0$ satisfies (A67) with (A68) and condition A.1 (3). Therefore, Equation (A74) is a general equation for g^s. Again, by letting $\eta_1 = \begin{cases} \ln(1+k), & \Pr = 0.5 \\ \ln(1-k), & \Pr = 0.5 \end{cases}$ and taking the limit, we can obtain a differential equation:

$$(g^s)' - (g^s)'' + 2a_3(e^s - 1) + 8a_8(e^s - s - 1) = const. \tag{A75}$$

Using $g^s(0) = 0$, g^s is represented as follows:

$$g^s(s) = a_9 s + a_{10}(e^s - 1) + 4a_8 s^2 + (8a_8 + 2a_3)se^s \tag{A76}$$

with additional constants a_9 and a_{10}. Substituting it into (A64) yields:

$$\begin{aligned} g(s, M_2, M_3) = a_1M_2 + a_2M_3 + (a_3M_2 + a_4M_3)(e^s - 1) + a_8(M_2 + aM_3 - 2s)^2 \\ + 4a_8(M_2 + aM_3)(e^s - 1) + a_9 s + a_{10}(e^s - 1) + (8a_8 + 2a_3)se^s \end{aligned} \tag{A77}$$

or

$$\begin{aligned} g(s, M_2, M_3) = d_1M_2 + d_2M_3 + d_3M_3e^s + d_4(M_2 + aM_3 - 2s)^2 + d_5(M_2 + aM_3 + 2s)e^s \\ + d_6 s + d_7(e^s - 1) \end{aligned} \tag{A78}$$

where $d_1 = a_1 - a_3$, $d_2 = a_2 - a_4$, $d_3 = a_4 - aa_3$, $d_4 = a_8$, $d_5 = a_3 + 4a_8$, $d_6 = d_9$ and $d_7 = d_{10}$. Then, substituting these into (A3) yields

$$\begin{aligned} d_4(4s_{1,1} + 2(m_2 + am_3 - \alpha_2 - a\alpha_3))(E[2\eta_1] + \alpha_2 + a\alpha_3) + (e^{s_{1,1}} - 1)(d_5(E[2\eta_1 e^{\eta_1}] \\ - \alpha_2 - a\alpha_3) - d_3\alpha_3) = 0 \end{aligned} \tag{A79}$$

Because s_1 is arbitrary, we have the following cases.

Condition A.2

(1) $d_3 = d_4 = d_5 = 0$

(2) $d_3 = d_5 = 0$ and $E[2\eta_1] + \alpha_2 + a\alpha_3 = 0$

(3) $d_4 = 0$ and $E[2\eta_1 e^{\eta_1}] - \alpha_2 - h\alpha_3$ with $h = a + d_3/d_5$.

Recall that $\mathrm{E}[e^{\eta_1} - 1] = 0, \alpha_k = E[f^k(\eta_1)]$ and $\frac{f^2(\eta_1)+af^3(\eta_1)}{\eta_1^2} \to 1$ for $\eta_1 \to 0$. Therefore, when condition A.2 (2) holds, we obtain $f^2(\Delta s) + af^3(\Delta s) = 2(e^{\Delta s} - \Delta s - 1)$. Next, condition A.2 (3) is equivalent to $d_3 = d_4 = 0$ with

$$E[2\eta_1 e^{\eta_1}] - \alpha_2 - a\alpha_3 = 0, \tag{A80}$$

which implies that

$$f^2(\Delta s) + af^3(\Delta s) = 2\left(\Delta s e^{\Delta s} - e^{\Delta s} + 1\right). \tag{A81}$$

Rearranging the above equations yields the equation and the condition of Proposition 3.1. This implies that Equation (14) is a candidate for a function with the aggregation property.

(Proof for the second statement)

Similar to Proposition 3.3, it is enough to show the SAP of Equation (14) holds. The SAP of the first four terms in the equation is obvious. Proofs for the 5th and 6th terms in this proposition are similar to those of the 10th and 11th terms in Proposition 3.1, respectively.

■

Appendix 2

Proof for Proposition 4.2

Let us set $S_t = S_{1,t} + aS_{2,t}$ and $X_t = (M_t^{(1)}, \ldots, M_t^{(L-1)}, 0)$ with $M_t^{(l)} = {}_t\kappa_l(S_T)$. Then, according to Fukasawa and Matsushita (2021),

$$E_0\left[\sum_{j=1}^{N} B_L\left(X_{t_j} - X_{t_{j-1}}\right)\right] = E_0[B_L(X_T - X_0, 0)] = \kappa_L^0(S_T). \tag{A82}$$

According to Equation (18), $M_t^{(l)}$ is decomposed to $\sum_{k=0}^{l} a^k \binom{l}{k} M_t^{(l-k,k)}$ for $M_t^{(l-k,k)} = {}^t\kappa_{l-k,k}(S_{1,T}, S_{2,T})$. In other words, the right hand side of Equation (A82) can be represented as

$$\sum_{k=0}^{L} a^k \binom{L}{k} {}^0\kappa_{L-k,k}(S_{1,T}, S_{2,T}). \tag{A83}$$

For convenience, we denote the summand of the left hand side of Equation (A82) as $B_L(\Delta X)$. By the definition of B_L, we can arrange it as follows.

$$\begin{aligned}
B_L(\Delta X) &= \frac{\partial^L}{\partial u^L}\left(\exp\left(\sum_{i=1}^{L-1} \Delta M^{(i)} \frac{u^i}{i!}\right)\right)\Bigg|_{u=0} = \frac{\partial^L}{\partial u^L}\left(\exp\left(\sum_{i=1}^{L-1}\sum_{j=0}^{i} a^j \binom{i}{j} \Delta M^{(i-j,j)} \frac{u^i}{i!}\right)\right)\Bigg|_{u=0} \\
&= \frac{\partial^L}{\partial u^L}\left(\exp\left(\sum_{i_0=1}^{L-1} \Delta M^{(i_0,0)} \frac{u^{i_0}}{i_0!} + a\sum_{i_1=1}^{L-1} i_1 \Delta M^{(i_1-1,1)} \frac{u^{i_1}}{i_1!} + O(a^2)\right)\right)\Bigg|_{u=0} \\
&= \frac{\partial^L}{\partial u^L}\left(\exp\left(\sum_{i_0=1}^{L-1} \Delta M^{(i_0,0)} \frac{u^{i_0}}{i_0!}\right)\right)\Bigg|_{u=0} \\
&\quad + a\sum_{i_1=1}^{L-1}\binom{L}{i_1} i_1 \Delta M^{(i_1-1,1)} \frac{\partial^{L-i_1}}{\partial u^{L-i_1}}\left(\exp\left(\sum_{i_0=1}^{L-1} \Delta M^{(i_0,0)} \frac{u^{i_0}}{i_0!}\right)\right)\Bigg|_{u=0} + O(a^2) \\
&= B_L\left(\Delta M^{(1,0)}, \cdots, \Delta M^{(L-1,0)}, 0\right) \\
&\quad + aL\sum_{i_1=1}^{L-1}\binom{L-1}{i_1-1} \Delta M^{(i_1-1,1)} B_{L-i_1}\left(\Delta M^{(1,0)}, \cdots, \Delta M^{(L-i_1,0)}\right) + O(a^2)
\end{aligned} \tag{A84}$$

Because a is arbitrary, the coefficient of a of the left hand side of Equation (A82) is equal to the coefficient of a of the right hand side of Equation (A82). Therefore, by Equations (A82), (A83) and (A84), we have

$$^{0}\kappa_{L-1,1}(S_{1,T}, S_{2,T}) = E_0\left[\sum_{j=1}^{N}\sum_{i_1=1}^{L-1}\binom{L-1}{i_1-1}\Delta M^{(i_1-1,1)}B_{L-i_1}\left(\Delta M^{(1,0)}, \Delta M^{(2,0)}, \cdots, \Delta M^{(L-i_1,0)}\right)\right]. \quad \text{(A85)}$$

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