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Partial hedging in credit markets with structured derivatives: a quantitative approach using put options

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Abstract

This study develops a novel method for mitigating credit risk through the use of structured derivatives, focusing in particular on the use of European put options as a strategic hedging tool. Inspired by the work of Merton (1974), our approach introduces the concept of default triggered by the stock price S_T breaching a predefined barrier B. By establishing a distributional equivalence between an existing default model and $\mathbb{P}(S_T < B)$ for a given time T, we demonstrate the potential for reducing the necessary capital allocation for a projected loss $X(T)$ by partially hedging with a European put option. We formulate and solve an optimization problem w.r.t. a specific risk measure to determine the optimal strike price for the option, and our numerical analysis confirms a reduction in the Solvency Capital Requirement (SCR) in markets with and without jumps. Our findings provide (insurance) companies with a pragmatic approach to mitigating losses while maintaining their current risk management framework.

Keywords Credit risk management, Equity derivatives, Partial hedging strategies, SCR reduction, Distance to default, Connection of debt and equity

Paper type Research paper

1. Introduction

In this paper, we explore a method for mitigating credit risk exposure through the strategic use of equity derivatives, drawing upon the foundational works by Merton (1974) and Black and Cox (1976) as our conceptual underpinning. We examine a company's balance sheet and denote by the variable V_t the company's assets at time $t \geq 0$. In alignment with fundamental accounting principles, these assets are financed by equity and debt. Equity is quantified as the product of the share price S_t and the total number of outstanding shares, which we denote by $N \in \mathbb{N}_{>0}$, i.e. $E_t = S_t N$. Debt D, on the other hand, aggregates all pending financial obligations with maturity T [1]. Thus, for fixed time T, the equity value is conceptualized as a call option on the company's assets with a strike price equal to its debt, formalized as $E_T = S_T N = (V_T - D)^+$, a notion initially posited by Merton (1974). The structural framework of Black and Cox (1976), on the other hand, introduced the concept of default as the moment the asset value falls below the debt level, equivalently when the equity value and consequently the share price drop to zero [2], expressed as:

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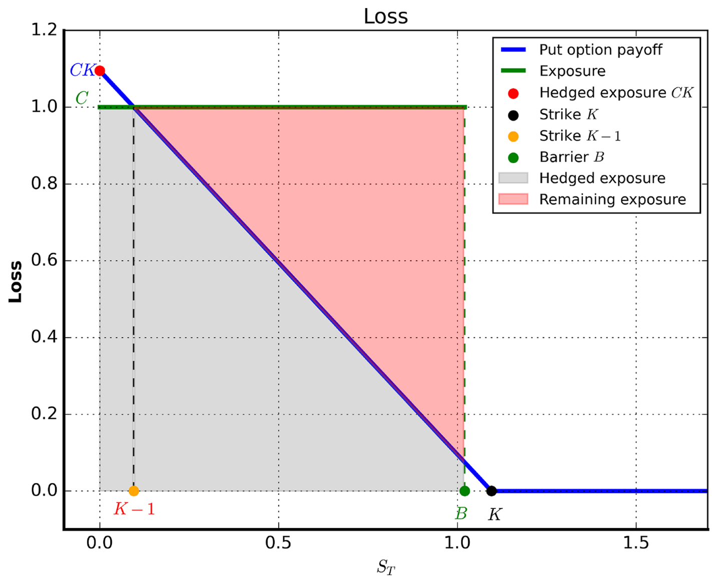
$$\tau = \inf\{t > 0 : V_t \le D\} = \inf\{t > 0 : E_t = 0\} = \inf\{t > 0 : S_t N = 0\} = \inf\{t > 0 : S_t = 0\}.$$

However, this definition may oversimplify the complexity of real-world defaults, which can precede the stock price reaching zero, influenced by factors like market delays, pending derivative contracts, or speculative trading activities (take Wirecard as an example: Even after default has been announced, trading in the stock remained active). Accordingly, we propose the establishment of an alternate, positive default threshold $B > 0$, wherein default is determined by the stock price's initial crossing of this predefined level. Both model definitions—the first involving a first-passage model over time and the second concerning a fixed time T—serve different purposes in various applications. We specify a company's credit exposure $C > 0$ to the at-risk entity, defined as the total outstanding debt maturing at $T > 0$. This places us within the framework established by Merton, focusing specifically on the time point T rather than considering the entire time span. Building upon that framework, we introduce the concept of loss X as a random variable functionally dependent on the share price at maturity S_T [3]:

$$X := f_B(S_T) = C\mathbb{1}_{\{S_T \le B\}}. \tag{1}$$

Evidently, the loss profile closely mirrors that of a digital put option based on S_T, which pays out the exposure amount $C > 0$. Although this financial instrument would serve as an ideal representation or hedge, it is not available in the market. Consequently, we employ European put options as the most viable alternative. Generally, put options offer only an approximation of the loss profile, illustrated in Figure 1.



Note(s): We note that costs are excluded in this graphic
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Figure 1.
Payoff profiles of loss and put option

In the depicted scenario, we set the exposure at $C = 1$. The green line represents the original loss, while the blue line illustrates the payout from a put option with a strike price of K. It is observable that incorporating a put option alongside the original exposure C yields a mitigated risk coverage: the area shaded in light blue indicates the risk offset by the specific put option, whereas the red-shaded area delineates the residual risk post the partial hedging with the put option. Thus, we intend to (partially) hedge the original loss X using the portfolio $a(P_K - P_{K,0})$, where

$$P_K = (K - S_T)^+$$

is the payout of the put option at maturity T and $a \in \mathbb{R}$ signifies the number of options purchased, with $P_{K,0}$ being the initial fair price of the put option. However, several questions still persist:

(1) Which option, respectively, which strike K [4] to choose?

(2) How many of such options should we buy, i.e. what is the optimal a?

(3) Does this offer any benefits for the (insurance) company as the purchase of these options comes at the cost of the premium?

To address the initial question, it is imperative to establish a criterion to optimize the decision-making process. We note that the approach outlined above implies a linear replicating portfolio in the put option; hence, as we aim for the best replication, we need to find an option that maximizes a linear fit with the original loss X. This leads to the classic Pearson's correlation as a maximizing criterion. As the model implicitly gives all other parameters and the estimation therein or the definition of the loss (maturity T of the option coincides with the maturity of the exposure), we aim to optimize for the strike K. Furthermore, due to the calculation rules of the correlation, we have:

$$Corr(X, P_K) = Corr\left(C\mathbb{1}_{\{S_T \leq B\}}, P_K\right) = Corr\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right).$$

Therefore, the problem can be formulated as follows:

$$\max_{K \in \mathbb{R}_+} \; Corr\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right). \tag{2}$$

Having established the strike, for which the corresponding option maximizes the linear fit with the original loss X, we then focus on how many such options we should buy. For this, we again formulate an optimization problem, but this time, we minimize the expected squared distance between the two portfolios dependent on the parameter $a \in \mathbb{R}$. This leads to the following optimization problem:

$$\min_{a \in \mathbb{R}_+} \; \mathbb{E}\left[(X - a(P_K - P_{K,0}))^2\right]. \tag{3}$$

As the strike is given by the previous optimization combined with the parameters determined by the market model for S_T, this optimization problem can be solved directly. These considerations are taken under the real-world measure $\mathbb{P}$. Hence, solving the optimization problem by basic calculations yields:

Lemma 1.1. **(Solution of** (3)**)** *The optimal solution* $a^* \in \mathbb{R}_+$ *for the optimization problem* (3) *is given by*

$$a^* = \frac{\mathbb{E}[X(P_K - P_{K,0})]}{\mathbb{E}\left[(P_K - P_{K,0})^2\right]},$$

where X is given in (1).

Given that the loss X and the payout of put options P_K are functions of the stock price S_T, the optimization problem (2) is well defined. Intuitively, selecting a higher strike K could enhance the fit; however, it implies increased costs. This necessitates addressing the subsequent question regarding cost considerations. According to Basel II regulations, (insurance) companies must maintain a designated reserve of capital, known as the Solvency capital requirement (SCR), to either fully or partially offset the potential claims outstanding. This regulatory capital requirement, stipulated under EU regulations, is intended to encompass 99.5% of the unexpected losses. Within the context of our analysis, the risk is denoted by X, allowing us to express the SCR for this risk as:

$$SCR_X := VaR_\alpha(X - \mathbb{E}[X]),$$

where α is customarily set at 99.5 per cent. Consequently, the feasibility of a (partial) hedge utilizing $a(P_K - P_{K,0})$ is contingent upon achieving a positive reduction in SCR, mathematically represented as:

$$SCR_X - SCR_{X-P_K} > P_{K,0}.$$

This inequality must be satisfied to address cost considerations [5]. Contrary to the literature, we do not use structural models to explain default, but assume to have a default model in place and aim to use this as a connection [6]. From a mathematical perspective, we introduce the default time of a company as the random variable τ, which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When engaged in CDS contracts, insurance companies typically serve as protection sellers, thereby assuming the credit risk associated with their counterparties. These contracts are quoted either through an upfront premium coupled with a fixed coupon or via a par spread, as indicated in Michielon *et al.* (2022). To dynamically convert between quoting conventions [7] it is assumed that the hazard rate Λ: $(0, +\infty) \mapsto (0, +\infty)$ defined by the default probability

$$\mathbb{P}(\tau \leq t) = 1 - \exp\left(-\int_0^t \Lambda(s)ds\right),$$

is constant (White, 2014). Hence, the random default time is assumed to follow an exponential distribution [8] ([Packham *et al.*, 2013; Appendix]), i.e.

$$\mathbb{P}(\tau \leq t) = 1 - e^{-\Lambda t}. \tag{4}$$

In this paper, we adopt the assumption that the marginal default distribution of τ is an exponential model. In summary, the strategy that is deployed in this paper comprises of the following steps:

S1 Fix a model for S_T.

S2 Find the parameters for the model for S_T and the necessary barrier B.

S3 Solve optimization problem (2) [9], calculate the optimal quantity a^* in Lemma 1.1.

S4 Obtain the loss profile to argue for a reduction in SCR.

The models under consideration include the traditional Black–Scholes model and enhanced jump-diffusion models featuring both log-normal and constant jump sizes. For all models discussed, we establish a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and introduce $\{W_t\}_{t\in(0,T)}$ to represent Brownian motion, $\{L_t\}_{t\in[0,T]}$ as a compound Poisson process (CPP) subordinator. Subsequent sections begin with a review of existing literature on default considerations in structural models and the strategies for partial hedging. After exploring the dynamics of $\{S_t\}_{t\in[0,T]}$ both with and without jumps, yielding unique maximuma of the correlations, we analyze the unexpected loss profile engendered by including the put option P_K to show **S4**. The discussion culminates with a conclusive summary.

2. Literature review

This paper contributes to two main areas of literature, i.e. *default prediction in structural models* and *partial hedging*.

2.1 Default prediction in structural models

There have been several approaches to assessing credit risk in the literature, most prominently reduced form and structural models. Statistical models have started with the Altman Z-Score (Altman, 1968) and have evolved into complex machine learning methods, including regression trees, neural networks, and many more (Siggelkow and Fernandez, 2023). In structural models (named by Duffie and Singleton (1999)), on the other hand, it is assumed that certain stochastic dynamics drive the value of a firm's assets and that a default occurs when the realization of this process is lower than the facial value of the firm's debt at its maturity. This approach was pioneered by Black and Scholes (1973) and Merton (1974). The authors assumed that the asset value follows a geometric Brownian motion, and the default probability can be calculated using the standard normal distribution. This model assumption has subsequently been extended to take into account various features, such as sovereign issuers (Gray *et al.*, 2019), the existence of multiple maturities for the firm's debt (Geske, 1977), or the possibility for the default to occur before the debt's maturity (Black and Cox, 1976; Leland and Toft, 1996); these latter models are often referred to as first passage or barrier models [10]. In Merton (1974), the author treated equity as a call option on the firm's asset value. Under this structural modeling approach, the corporate bond credit spread becomes a function of financial leverage and firm asset volatility. The financial leverage then links equity to debt and relates firm volatility to equity volatility. In Merton (1976), the author recognized the direct impact of corporate default on the stock price process and assumes that the stock price jumps to zero and stays there upon the random arrival of a default event. Black introduced the so-called DD, building on this connection between debt and equity. The measure estimates the probability that the value of the firm/asset falls beneath the value of the debt. After having been clarified in Crosbie (2003), many researchers have analyzed this measure and concluded that it is a main covariate to asses and predict financial distress (Duffie *et al.*, 2009), as well as to explain CDS spreads (Bai and Wu, 2011), pure credit contract prices (Carr and Wu, 2011), or equity returns (Vassalou and Xing, 2004). The authors of Jessen and Lando (2015) showed that despite the simplifying assumptions that underlie its derivation, the DD measure under the geometric Brownian motion model has proven empirically to be a strong predictor of default. They further concluded that DD generally ranks firms' default probabilities, even if the underlying model assumptions are altered, i.e. the geometric Brownian motion assumption is relaxed. The authors of Bharath and Shumway (2008) provided evidence showing that the functional form of Merton's DD model makes it useful and important for predicting defaults. They compared the model to a "naive" alternative, which uses the functional form suggested by the Merton model but does not solve the model for an implied probability of default. They concluded that while the Black DD model does not produce a sufficient statistic for the probability of default, its functional form is useful for forecasting. Building on that idea are the authors of Chen and So (2014), who investigated whether the default predictability of the Merton DD model would be affected by considering investors' ambiguity aversion and concluded that this model performs better than the naive model in Bharath and Shumway (2008). In Friewald *et al.* (2014), the authors investigate the relationship between credit risk premia, derived from Credit Default Swap (CDS) spreads, and equity returns. The paper uses firm-specific measures and arbitrage-free term structure models to estimate credit risk premia, demonstrating that CDS spreads provide equity-relevant information distinct from common distress risk measures. The study finds that firms with higher credit risk premia tend to have higher default probabilities and that credit risk premia are informative of equity returns, particularly during financial crises when CDS spreads exhibit time-varying risk premia. While the Brownian assumption infers a tractable model, numerous authors pointed out that stock returns are not normally distributed, significantly limiting the model's use in practice (Brambilla *et al.*, 2015). Moreover, the estimates of the probability of default can be biased downwards, exposing the banks to the possibility of undercapitalization and systematic shocks. Furthermore, the Brownian hypothesis has been criticized for producing an almost zero default probability for short maturities: if this were true, then short-term bonds should have close to zero credit spread, which is typically not the case, as first noted in Philip Jones *et al.* (1984) (see also subsequent discussions such as Demchuk and Gibson (2006)). This underestimation of short-term default probabilities and theoretical credit spreads is known as the credit spread puzzle. It has been evidenced many times and for every ranking, from high yield to investment grade issuers (see a recent overview in Nozawa *et al.* (2019)) and even sovereign issuers (see Duyvesteyn and Martens (2015) and references therein). The plausible explanation for this discrepancy is that other factors can influence credit spreads, such as taxes, liquidity premia, and jump risk (Bai *et al.*, 2020). These arguments led researchers to extend the Brownian model. The natural idea is to introduce jumps in the asset dynamics to better capture short-term defaults occurring after a sudden drop in the value of a company's assets following, for instance, earning announcements, central bank meetings, or major political events. Jump processes can be of two kinds: jump-diffusion or pure jumps processes, and the formalism of Lévy processes conveniently describes both. Jump diffusion processes were introduced in structural credit risk modeling in Zhou (1997) by adding a Poisson process whose discrete jumps are normally distributed to the usual diffusion process to materialize sudden changes in the firm's assets value; other distributions for the magnitude of the Poisson jumps have subsequently been considered, such as negative exponential distributions, leading to higher short term probabilities and more realistic credit spread curves. Self-exciting Hawkes processes have also been introduced in Ma and Xu (2016), a model for which an analytical formula for the equity value has been recently derived in Pasricha *et al.* (2021). Pure jump processes allow for an interpretation in terms of business time (differing from the operational time) or the possibility for jumps to occur arbitrarily often on any time interval; such processes were introduced in the construction of credit risk models in the late 2000s and early 2010s, notably in Madan and Schoutens (2008) for one-sided processes (i.e. featuring downward jumps only) and from the point of view of first passage models, and in Luciano (2010) for double-sided processes, with CDS-based calibrations. Extensions to multivariate processes have also been studied in Marfe (2012). The authors of Aguilar *et al.* (2021) extended the literature on DD modeling with pure jump processes. They showed that including jump processes is necessary and that jump-diffusion processes are outperformed in default prediction because short-term default probabilities are not underestimated. The authors of Jovan and Ahčan (2017) extended the Brownian model by allowing for NIG distributed returns. By applying their results to the Ljubljana stock exchange, they found that the probability of default estimates using the Brownian model is biased. In contrast, the estimates from the NIG Merton model are robust. In Benos and Papanastasopoulos (2007), the authors enriched the DD metric with financial ratios and accounting variables into a hybrid model. They concluded that the combination improves both the in-sample fit of credit ratings and the out-of-sample predictability of defaults. The authors of Guo and Li (2022) recently analyzed a stressed version of the DD model by controlling for raw default probability and failure beta. They showed that the stressed DD can explain credit default swap spreads and rating variations. In Ibañez (2023), the authors most recently demonstrated that default events in endogenous credit-risk models, initially defined by low asset values, can also be characterized by low equity prices and negative net cash flows, measured through the volatility-adjusted DD ratio. As mentioned, default is associated with a low asset value under the Black/Merton approach. According to Ibañez (2023), this corresponds to a low equity price/large and negative cash flow, and hence, the default barrier can be chosen differently (in line with the findings in Black and Cox (1976), Leland and Toft (1996)). An alternative to Merton are Leland-type models (Leland, 1994), where the default threshold is taken endogenously by equity holders who maximize the value of their equity stake. The default is an American put associated with a low asset value/large and negative flow. The authors of Carr and Wu (2011) proposed a simple and robust link between equity out-of-the-money American put options and a credit insurance contract on the same reference company. This means it is natural to consider put options when considering credit risk. These Leland-type models infer that the default event might be strategic, as both equity holders and debt holders can have incentives to induce or force bankruptcy well before the equity value completely vanishes. In Gordy and Carey (2007), the authors find empirical support for active strategic behaviors from private debt holders in setting the endogenous asset value threshold below which corporations declare bankruptcy. In particular, they find that private debt holders often find it optimal to force bankruptcy well before the equity value vanishes. This again implies that it is natural to consider a barrier different from the value of the debt in these passage models, which aligns with our approach in this paper.

2.2 Partial hedging

Option pricing theory postulates that most contingent claims can be hedged, provided the starting capital is at least equal to the claim's fair value in complete markets or the minimum super-hedging price in incomplete markets. Nonetheless, the expenses associated with perfect hedging are often prohibitively high for practical purposes, leading to the emergence of partial hedging strategies. In these strategies, investors use initial capital below the cost of perfect hedging to devise strategies to reduce their potential losses according to specific risk metrics. Föllmer and Leukert have been instrumental in advancing the concept of optimal partial hedging, exploring areas such as quantile hedging and efficient hedging within semimartingale financial market models, as detailed in Föllmer and Leukert (1999) and Föllmer and Leukert (2000), respectively. They offer straightforward solutions for complete markets utilizing the classical Neyman-Pearson lemma and provide theoretical solutions for incomplete markets through the convex duality method. The exploration of partial hedging has also extended into more complex market scenarios. In Cvitanić and Spivak (1999), the authors delved into quantile hedging in contexts where information is partial and where market influence by large investors is significant. In Nakano (2011), the author addressed the optimization of quantile hedging and efficient hedging strategies for claims characterized by a single default time using linear loss functions. Furthermore, in Melnikov and Nosrati (2017), the authors examined various partial hedging approaches and their utility in the pricing and hedging insurance contracts, showcasing the broad applicability of these strategies beyond traditional financial markets. When devising partial hedging strategies, a critical factor for investors is the selection of the risk criterion. The concept of risk measures was extensively examined in Artzner *et al.* (1999), which laid the foundation for their application in the valuation and hedging of contingent claims, as demonstrated in Xu (2006). The optimal portfolio is identified in this context by minimizing a convex risk measure. Additionally, the literature includes significant contributions such as the study on Law-Invariance risk measures (Bernard *et al.*, 2015) and the exploration of risk measures derived from distortion functions (Madan and Schoutens, 2016), further enriching the field's understanding of risk assessment methodologies. In financial institutions, VaR and CVaR are the most commonly used risk measures, with a long list of references. For instance, the authors of Melnikov and Smirnov (2012) studied partial hedging w.r.t. the CVaR where the semi-explicit solution of the optimal CVaR hedging problem in complete markets was given. In Cong *et al.* (2013), the authors discussed VaR-based optimal hedging, while in Cong *et al.* (2014), the CVaR-based optimal hedging problem was solved without the restriction regarding the completeness of markets. In Melnikov and Wan (2022), the authors analyzed the more general tail risk measure named RVaR, which was applied to hedging problems in Cont *et al.* (2010) and Embrechts *et al.* (2018). They concluded that the optimal hedging strategy is a knock-out call option written on the risk.

2.3 Own contributions

In this paper, we present a method for (insurance) companies to strategically manage credit risk via derivatives, notably European put options. Our contributions to the literature are manifold:

(1) Deviating from conventional approaches in the literature, which predominantly employ structural models for modeling defaults directly, our approach is based on the premise of a predefined default distribution, $F(t)$, $t > 0$. This variation facilitates the integration of our findings into the risk management frameworks of (insurance) firms, offering a dynamic mechanism for credit risk mitigation.

(2) We define the potential loss as exposure (for instance, the sum of outstanding invoices) with an assigned maturity T. Imposing equality in the distribution in T, i.e.

$$F(T) \overset{!}{=} \mathbb{P}(S_T \leq B),$$

we establish and resolve an optimization problem to identify the European put option [11] that maximizes the correlation.

(1) Additionally, we demonstrate that applying this European put option solves the optimization problem with a unique maximum and substantially reduces the capital requirement of insurance companies.

To conclude, the insights derived from this paper provide (insurance) companies with the strategy to maintain risk within their operational framework while concurrently mitigating potential losses by acquiring European put options. The outcomes of our research directly furnish the optimal strike price for these options. We do not claim that our connection between the credit and equity world is novel, as numerous works (e.g. Friewald *et al.* (2014)) have previously focused on such a connection. However, while Friewald *et al.* (2014) focuses on the general connection between credit and equity markets, they do not directly incorporate a predefined risk model and distributional equivalence at maturity, T, a gap we aim to fill.

Furthermore, our method allows for practical application by companies, as it enables direct integration into daily risk management practices through numerical analysis of Solvency Capital Requirement (SCR) reduction. Additionally, Figure 2 summarizes the strategy of this paper and highlights our contributions.

3. Black–Scholes model

The first model we consider for S_T is the Black–Scholes model. We therein assume that the dynamics can be explained by the following SDE:

$$dS_t = S_t(\mu dt + \sigma dW_t), S_0 > 0,$$

where $\mu, \sigma > 0$ are drift and standard deviation of the process, respectively. Solving this SDE utilizing Itô for the maturity time in question leads to

$$S_T = S_0 e^{\left(\mu - 1/2\sigma^2\right)T + \sigma W_T}$$

and we have $W_T \sim \mathcal{N}(0, T)$. This concludes the model selection phase outlined in **S1**. The subsequent phase involves defining the model's parameters and determining the appropriate

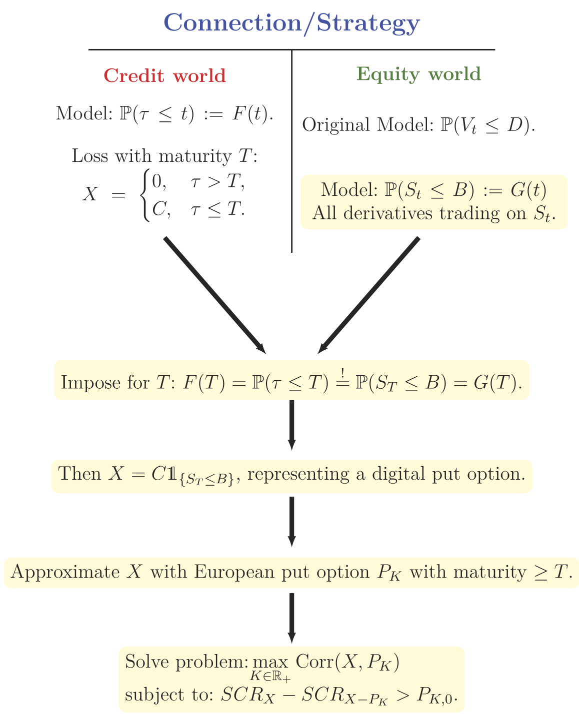


Figure 2.
Graphical representation of the approach deployed throughout this paper

Note(s): τ represents the default time of the at-risk company obtained with the pre-existing default model. The steps filled in yellow are the novel steps developed throughout this paper
Source(s): The figure is provided by the author

barrier level. It is at this juncture that the relevance of our previously established default model becomes apparent:

$$\mathbb{P}(\tau \leq t) = 1 - e^{-\Lambda t} = F(t).$$

Then, we let $\boldsymbol{\theta} = (\mu, \sigma)$ be the vector of model parameters, $G(t) := \mathbb{P}(S_t \leq B)$ and use a least squares estimation in the context of distributions, i.e. we minimize $\mathcal{L}^2(F, G)$ w.r.t. the parameters [12]:

$$(\widehat{\mu}, \widehat{\sigma}, \widehat{B}) = \arg\min_{\boldsymbol{\theta}, B} \frac{1}{2} \|F - G\|_2^2 = \arg\min_{\boldsymbol{\theta}, B} \frac{1}{2} \int_0^\infty (F(t) - G(t))^2 dt.$$

For numerical reasons, we introduce a set of discrete time points $t_0 < t_1 < \ldots < t_N$ [13] and consider the discretized version of this continuous setting, i.e.

$$(\widehat{\mu}, \widehat{\sigma}, \widehat{B}) = \arg\min_{\boldsymbol{\theta}, B} \frac{1}{2} \sum_{i=0}^{N} \left(1 - e^{-\Lambda t_i} - \mathbb{P}\left(S_{t_i} \leq B\right)\right)^2, \tag{5}$$

where N is the number of time points we use. In the Black–Scholes model context, we can explicitly calculate the default probability. This quantity is given by:

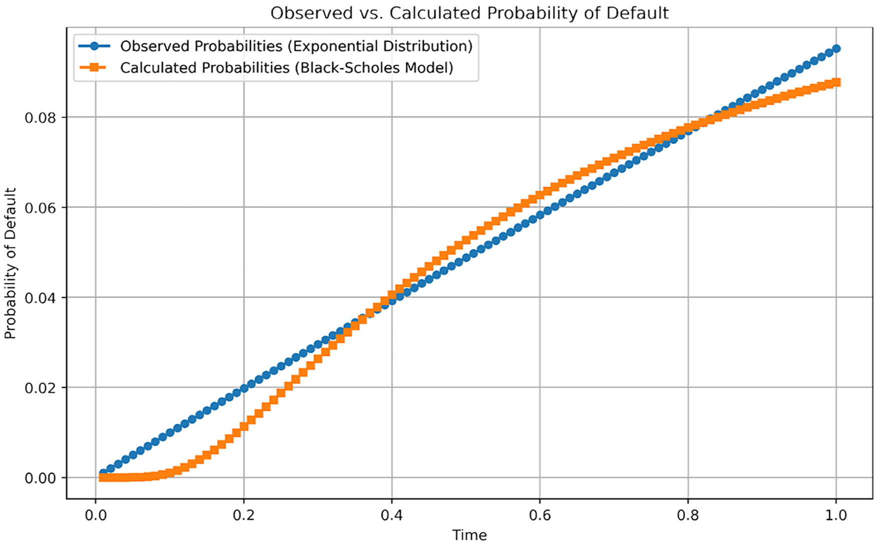
$$\mathbb{P}(S_t \leq B) = \boldsymbol{\Phi}\left(\frac{\ln(B/S_0) - (\mu - 1/2\sigma^2)t}{\sigma\sqrt{t}}\right). \tag{6}$$

Hence, the objective of (5) is to identify the model parameters and the barrier B such that the above normal distribution closely aligns with the predetermined exponential distribution over time while ensuring distributional equivalence for fixed T. An illustrative example of such an alignment is provided in the subsequent graphical representation, marking the completion of **S2**.

It becomes evident that for low default probabilities, as displayed in Figure 3a, the fit between the two distributions is more accurate than for higher default probabilities shown in Figure 3b. Additionally, the results exhibit similar values for both the drift and the volatility, despite differing default barriers $\widehat{B}$. Since this barrier must be crossed for a default to occur, it is more likely with $\widehat{B} = 0.97$, given that the initial value is $S_0 = 1$. Therefore, a higher barrier aligns with a higher default intensity, which is logical. Moreover, since our focus is on distributional equivalence at a specific time point T, it is crucial for the distributions to intersect at this time point, as demonstrated in both scenarios. For further numerical analysis, we concentrate on the estimation parameters from Figure 3b, corresponding to the higher default probability. Assuming accurate parameter estimation, we can rearrange (6) for the exposure maturity $T > 0$ to obtain,

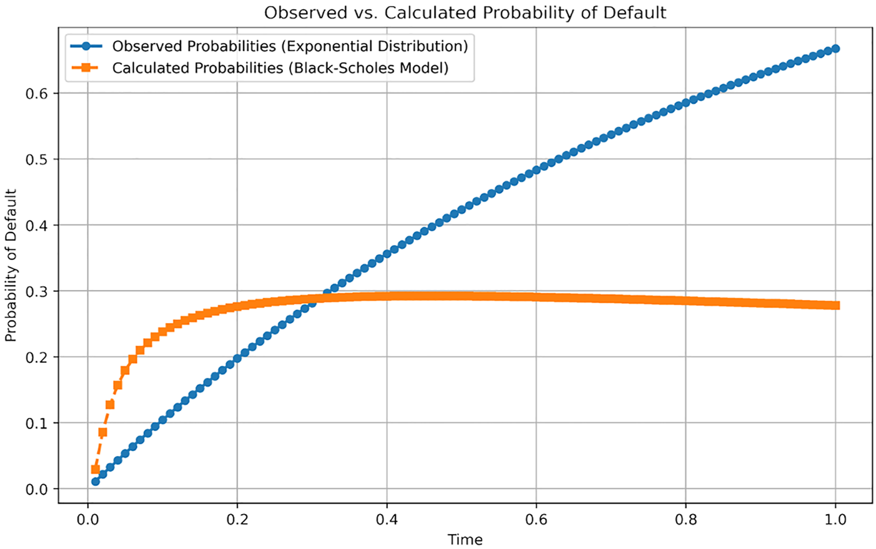
$$B = S_0 e^{\widehat{\sigma}\sqrt{T}\Phi^{-1}\left(1 - e^{-\Lambda T}\right) + \left(\widehat{\mu} - 1/2\widehat{\sigma}^2\right)T} > 0, \tag{7}$$

where $\widehat{\sigma}, \widehat{\mu}$ are the estimated parameters from (3b) [14]. As motivated in the introduction of this paper, we aim to identify the strike price K that maximizes the correlation between the default indicator process and the associated put option. Naturally, we consider this question under the real-world measure $\mathbb{P}$ and the dynamics (3). Recall the payout function of the put from (1).



Parameter estimation for $\Lambda = 0.1$, $S_0 = 1, T = 0.3$. This estimation results in $(\hat{\mu}, \hat{\sigma}, \hat{B}) = (0.07, 0.15, 0.87)$

(a)



Parameter estimation for $\Lambda = 1.1$, $S_0 = 1, T = 0.3$. This estimation results in $(\hat{\mu}, \hat{\sigma}, \hat{B}) = (0.07, 0.15, 0.97)$

(b)

Source(s): The figure is provided by the author

Figure 3.
Comparison of parameter estimations under different Λ values for the Black–Scholes model

Theorem 3.1. **(Correlation in the Black–Scholes model)** *In the Black–Scholes model with given parameters $\sigma, \mu > 0$, $\mathbb{P}$-dynamics as given in* (3)*, maturity time $T > 0$, and barrier B obtained via* (7)*, we have*

$$Corr\left(\mathbb{1}_{\{ST\le B\}}, P_K\right) = \begin{cases} \dfrac{1}{f(P_K)} e^{-\Lambda T}\mathbb{E}[P_K] & \text{for } K \le B, \\ \dfrac{1}{f(P_K)}\left(\left(1-e^{-\Lambda T}\right)\left(K-\mathbb{E}[P_K]\right)\right. \\ \left. -S_0 e^{\mu T}\Phi\left(\Phi^{-1}\left(1-e^{-\Lambda T}\right)-\sigma\sqrt{T}\right)\right) & \text{for } K > B, \end{cases}$$

where $f(P_K) = \sqrt{e^{-\Lambda T}(1-e^{-\Lambda T})(\mathbb{E}[P_K^2]-\mathbb{E}[P_K]^2)}$. Further, we can conclude that there is a unique K^* maximizing the correlation.

Example 3.2. **(Illustration of correlation including its derivative)** *In the analysis of the correlation function and its derivative, which is divided into two segments, we employ distinct colorations for each segment within our graphical representation. In this numerical example we utilize $\Lambda = 1.1$, and the parameter estimation illustrated in* Figure 3b*, obtaining the values $(\widehat{\mu}, \widehat{\sigma}, \widehat{B}) = (0.07, 0.15, 0.97)$. With* (7)*, we can explicitly calculate B in this model and obtain, as expected, $B = \widehat{B} = 0.97$* [15]*. The selected time horizon is $T = 0.3$ years, corresponding to approximately four months, deemed a practical duration for assessing exposure. The analysis is displayed in* Figure 4.

In addition to solving the optimization problem (2) explicitly, we can also simulate the optimal quantity as displayed in Lemma 1.1. Figure 5 displays the optimal quantity for the strike price established in Example 3.2.

4. Jump-diffusion model with lognormal jumps

As discussed in the literature review, while pioneering, the Black–Scholes model exhibits certain limitations (e.g. underestimation of short-term default probabilities, negligence of jumps) that necessitate the inclusion of jumps to more accurately reflect market dynamics. In Merton (1976), the author introduced the concept of jumps characterized by a lognormal distribution with parameters μ_J, σ_J^2, to address these limitations. The author derived and related the SDE dynamics to the Black–Scholes model. For details, we refer the reader to the source. However, we end up with the following conditional dynamics:

$$S_t^{(k)} = S_0^{(k)} e^{\left(\mu - 1/2\sigma_k^2\right)t + \sigma_k W_t}, \tag{8}$$

where $S_t^{(k)}$ is equal to S_t conditioned on the number of jumps $\{N_t = k\}$ and σ_k, $S_0^{(k)}$ are adjusted standard deviation and a modified initial value. The construction again implies mutual independence between N_t and W_t, and a fixed T represents the exposure's maturity. This concludes **S1**. The subsequent step **S2** implies further complexity compared to the Black–Scholes model as the introduction of jumps results in non-continuous paths of S_t. Comparing this with a continuous distribution such as the exponential is non-trivial. Furthermore, more parameters must be estimated, as the jump parameters must also be considered. Therefore, we leave this for further research, as parameter estimation for Lévy processes is a field of

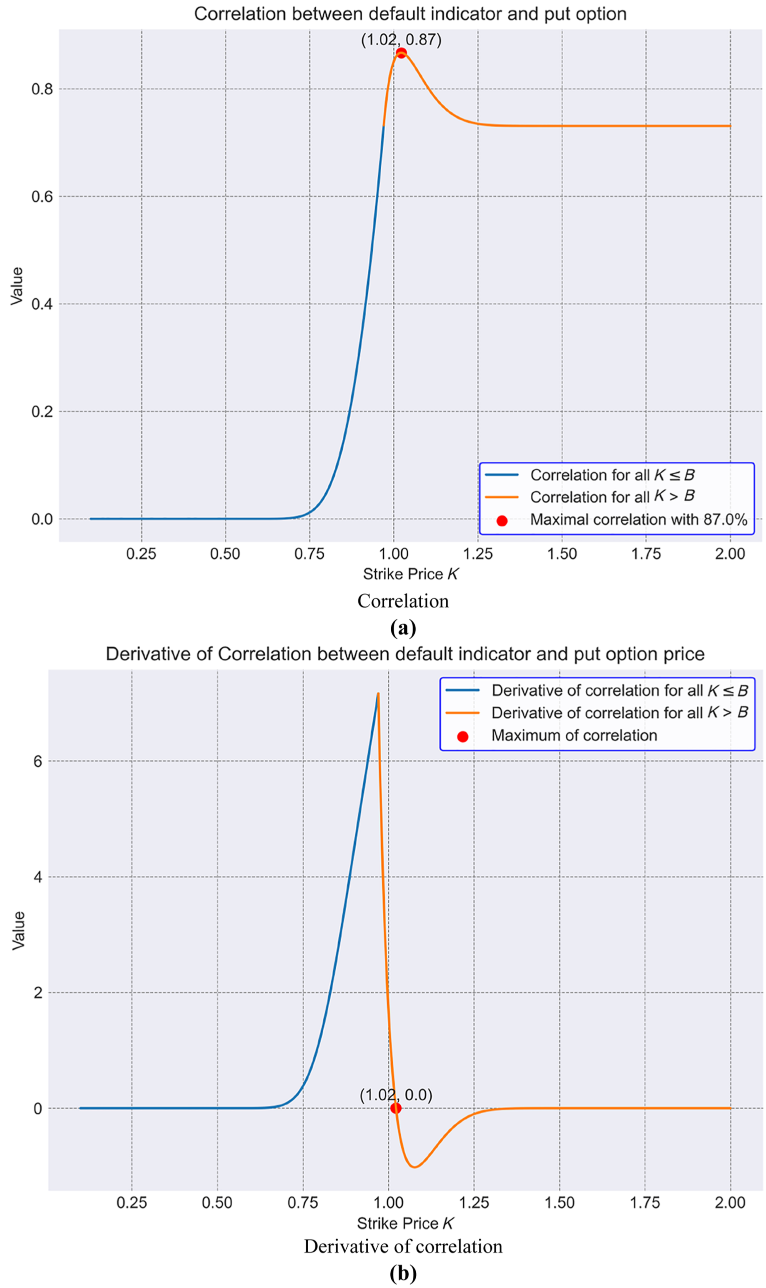


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Figure 4.
Numerical illustration of the correlation and the derivative of correlation with parameters $S_0 = 1$, $\widehat{\mu} = 0.07$, $T = 0.3$, $\widehat{\sigma} = 0.15$, $\Lambda = 1.1, B = \widehat{B} = 0.97$

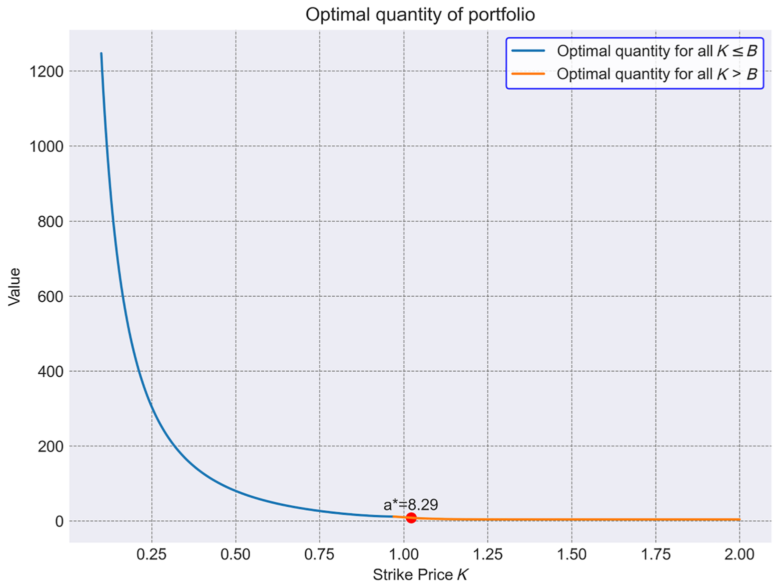

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Figure 5. Numerical demonstration of optimal quantity utilizing parameters $S_0 = 1, \widehat{\mu} = 0.07, T = 0.3, \widehat{\sigma} = 0.15, \Lambda = 1.1, B = \widehat{B} = 0.97, K^* = 1.02$, resulting in $a^* = 8.29$

theory in itself. However, we can formulate a condition for the barrier B, similar to the Black–Scholes model. Therefore, we impose that for the discrete-time point T (maturity of exposure and, hence, option), we have equality in the distribution, i.e.

$$\begin{aligned}\mathbb{P}(S_T \leq B) &= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\mathbb{P}(S_T \leq B \mid N_T = k) \\ &\overset{(8)}{=} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\mathbb{P}\left(S_0^{(k)} e^{(\mu - 1/2\sigma_k^2)T + \sigma_k W_T} \leq B\right) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\mathbb{P}\left(\frac{1}{\sqrt{T}} W_T \leq \frac{\ln\left(\frac{B}{S_0^{(k)}}\right) - (\mu - 1/2\sigma_k^2)T}{\sigma_k\sqrt{T}}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\Phi\left(\frac{\ln\left(\frac{B}{S_0^{(k)}}\right) - (\mu - 1/2\sigma_k^2)T}{\sigma_k\sqrt{T}}\right) \\ &=: \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\Phi(d(B,k)) \overset{!}{=} 1 - e^{-\Lambda T}.\end{aligned} \tag{9}$$

Diverging from the Black–Scholes model, the introduction of jumps eliminates the availability of a closed-form solution, necessitating numerical methods for root finding. It's important to note that the probability of observing k jumps over a period T, given by

$\mathbb{P}(N_T = k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$, becomes increasingly negligible for larger values of k, especially over short time horizons where the likelihood of numerous jumps is limited. This characteristic significantly simplifies the numerical root search process, as it allows the search to be constrained to a manageable range, such as up to $k = 5$.

In the visualization, the default probability is depicted in orange, while the probability of the stock price S_T breaching the barrier level B is illustrated in blue, with both probabilities viewed as functions of B. This graphical representation facilitates the identification of a unique barrier level B at which these two probabilities agree, supporting the previous argument. This concludes **S2** for this jump-diffusion model. Next, we reintroduce the put option as

$$P_K = (K - S_T)^+ = \sum_{k=0}^{\infty} \mathbb{P}(N_t = k)\left(K - S_0^{(k)} e^{\left(\mu - 1/2\sigma_k^2\right)T + \sigma_k W_T}\right)^+$$

$$:= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) P_{BS}\left(S_0^{(k)}, \sigma_k, K\right).$$

Given the adjusted parameters, it's crucial to understand that the inside of the function equates to a European put option within the Black–Scholes framework, written on strike K. Henceforth, this formula is denoted as $P_{BS}\left(S_0^{(k)}, \sigma_k, K\right)$, where $S_0^{(k)}$ and σ_k are the initial stock price and volatility adjusted for k jumps, respectively. In subsequent discussions, whenever we refer to P_K, it specifically denotes the put option priced in the context of this jump-diffusion market.

Theorem 4.1. **(Correlation in the Merton jump-diffusion model)** *In the Merton jump-diffusion model with lognormal jumps and given parameters σ, μ, μ_J, $\sigma_J > 0$, (conditional) $\mathbb{P}$-dynamics as given in* (8) *and barrier B obtained via* (9)*, we have*

$$Corr\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) = \frac{1}{f(P_K)} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) h(K, k),$$

where

$$h(K,k) = \begin{cases} e^{-\Lambda T}\mathbb{E}\left[P_{BS}\left(S_0^{(k)}, \sigma_k, K\right)\right], & K \le K_k^*, \\ \left(1 - e^{-\Lambda T}\right)\left(K - \mathbb{E}\left[P_{BS}\left(S_0^{(k)}, \sigma_k, K\right)\right]\right) \\ \qquad - S_0^{(k)} e^{\mu T}\Phi(d(B,k)) - \sigma_k\sqrt{T}\Big), & else \end{cases}$$

for $K_k^* = S_0^{(k)} e^{d(B,k)\sigma_k\sqrt{T} + (\mu - 1/2\sigma_k^2)T}$ and $f(P_K) = \sqrt{e^{-\Lambda T}(1 - e^{-\Lambda T})(\mathbb{E}[P_K^2] - \mathbb{E}[P_K]^2)}$. Further, we can conclude that there is a unique $\widehat{K}^*$ maximizing the correlation above.

Example 4.2. **(Correlation example including derivative)** *The selected time horizon for this numerical analysis is $T = 0.3$ years, corresponding to approximately four months, deemed a practical duration for assessing exposure. Furthermore, we chose the same parameters as for the Black–Scholes example to draw comparisons. The barrier B was estimated via the root search as depicted in Figure 6. The analysis again confirms the expectations as a unique maximum is found. In contrast to the Black–Scholes model, however, the correlation function is not piecewise. Figure 7 displays the numerical example.*

Additionally, Figure 8 displays the optimal quantity from Lemma 1.1 for the strike price established in Example 4.2.

5. Merton jump-diffusion model with constant negative jumps

The preceding section explored the incorporation of lognormal jumps as an augmentation to the Black-Scholes framework. This section draws inspiration from the Lévy factor portfolio default model outlined in Mai and Scherer (2017). Within that context, we introduce a Lévy subordinator, L_t, characterized as being of the Compound Poisson Process (CPP) type with a constant jump size $\theta_{credit} > 0$, expressed as

$$L_t = \theta_{credit} N_t,$$

where N_t denotes a CPP. Accordingly, a firm's default is the initial time L_t surpasses a unit-exponential threshold. Conceptually, an increase in L_t (presuming the absence of drift) diminishes the gap to the default threshold, thus potentially elevating the default risk. If we assume that this specific process also influences the dynamics in a structural framework, we can integrate this notion seamlessly into this paper by suggesting that a jump in the process

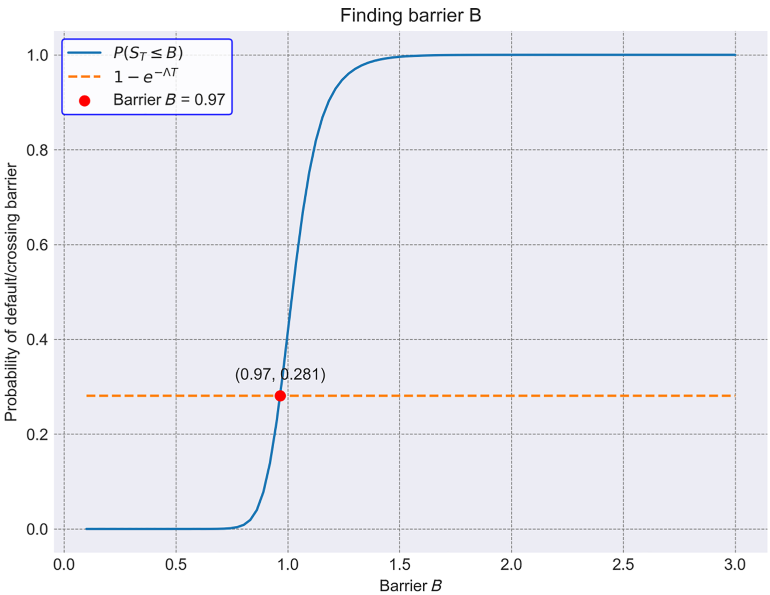

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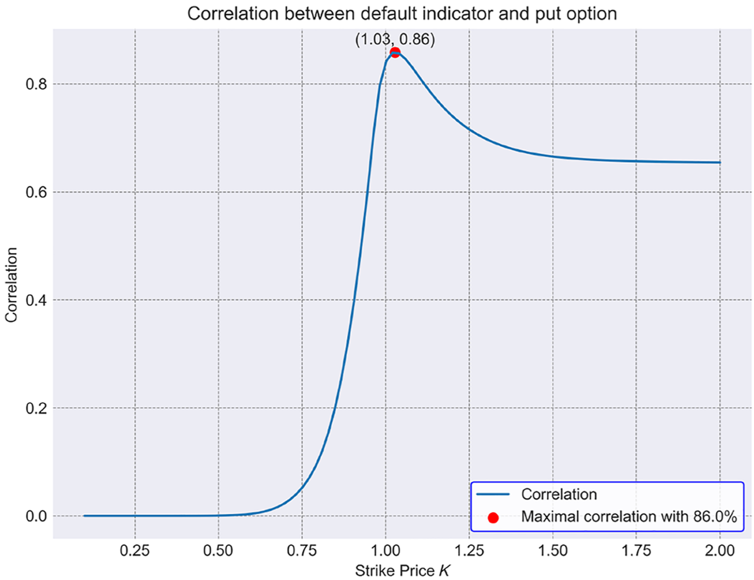
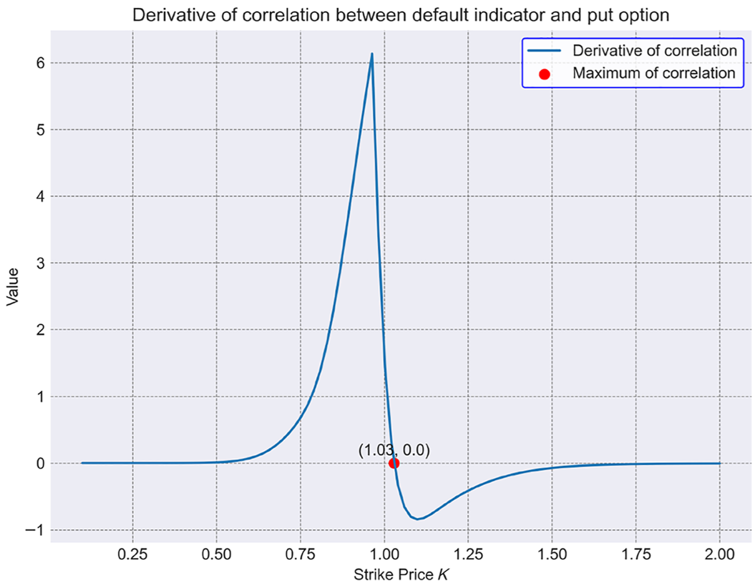
Figure 6.
Numerical root search for B with parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $\mu_J = 0.02$, $\sigma_J = 0.1$



Correlation

(a)



Derivative of correlation

(b)

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Figure 7.
Numerical illustration of the correlation function and its derivative for the parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $\mu_J = 0.02$, $\sigma_J = 0.1$, $B \approx 0.97$

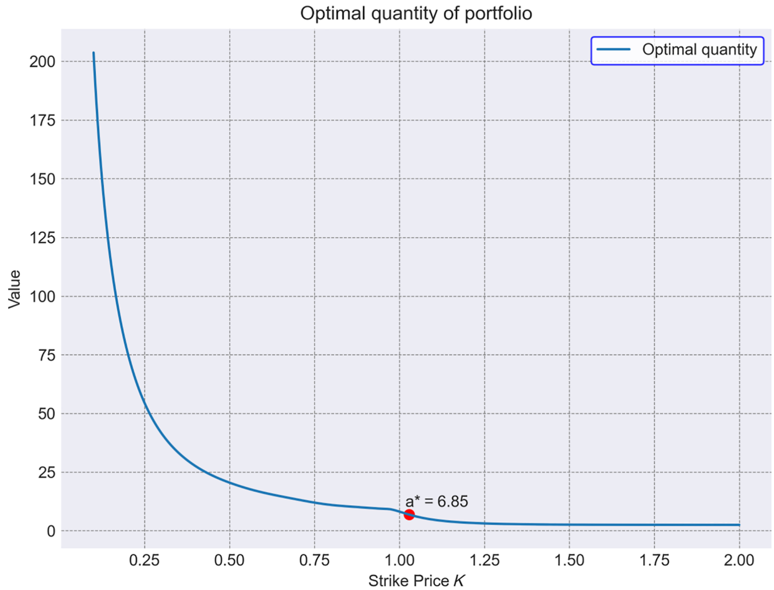

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Figure 8. Numerical demonstration of optimal quantity utilizing parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $\mu_J = 0.02$, $\sigma_J = 0.1$, $B \approx 0.97$, $\widehat{K}^* = 1.03$, resulting in $a^* = 6.85$

N_t signifies a detrimental impact on the asset value S_t, typically correlating with the advent of adverse news. This rationale motivates the examination of the subsequent SDE:

$$dS_t = S_t(\mu dt + \sigma dW_t + (1 - \theta_{asset})dN_t). \tag{10}$$

This represents a jump-diffusion model characterized by exclusively constant negative jumps $0 < \theta_{asset} < 1$ [16].

Lemma 5.1. **(Solution of SDE)** *In this model, the solution of the SDE* (10) *is given by*

$$S_T = S_0 \exp\{(\mu + \lambda(1 - \theta_{asset}) - 1/2\sigma^2)T + \sigma W_T\}(1 - \theta_{asset})^{N_T}, S_0 > 0.$$

We see that by conditioning on the number of jumps k and introducing an adjusted starting value $S_0^{(k)} = S_0(1-\theta_{asset})^k e^{\lambda(1-\theta_{asset})}$, we are again situated in the Black–Scholes world. The mutual independence of N_t and W_t follows again by construction. The option and exposure's maturity time T is set to coincide, ensuring that the distributions match at this specific point:

$$\begin{aligned}\mathbb{P}(S_T \le B) &= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\Phi\left(\frac{\ln\left(\frac{B}{S_0^{(k)}}\right) - (\mu - 1/2\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &:= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\Phi(d(B,k)) \overset{!}{=} 1 - e^{-\Lambda T}.\end{aligned} \tag{11}$$

In congruence with the last section, a numerical root search needs to be deployed, and by simulating for $k = 5$, we obtain the following graphic:

We again note that parameter estimation (as in the lognormal jump-diffusion model) is left for further research. The default definition on the credit side of this model is driven by the Lévy subordinator $\theta_{credit}N_T$, while on the structural side, it is influenced by the dynamics of S_T, which also depends on a process N_T. We assume that these processes are the same to enable meaningful comparisons. Contrary to the asset side, the credit side yields the possibility to calculate the jump size θ_{credit} explicitly.

Lemma 5.2. **(Choice of jump size $\boldsymbol{\theta_{credit}} > 0$)** *In this specific model, we have*

$$\theta_{credit} := -\ln\left(1 - \frac{\Lambda}{\lambda}\right).$$

Therefore, we need to ensure $\Lambda < \lambda$. More on this condition can be found in Example 5.4.

This concludes **S1** and **S2** for this jump-diffusion model. By using the adjusted starting value $S_0^{(k)}$ from above, we reintroduce the put option as

$$\begin{aligned} P_K = (K - S_T)^+ &= \sum_{k=0}^{\infty} \mathbb{P}(N_t = k)\left(K - S_0^{(k)} e^{\left(\mu - 1/2\sigma^2\right)T + \sigma W_T}\right)^+ \\ &:= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) P_{BS}\left(S_0^{(k)}, \sigma, K\right). \end{aligned}$$

Theorem 5.3. **(Correlation in the jump-diffusion model with constant negative jumps)** *In the jump-diffusion model with constant negative jumps* $0 < \theta_{asset} < 1$*, given parameters σ, $\mu > 0$ and barrier B obtained via* (11)*, we have*

$$Corr\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) = \frac{1}{f(P_K)} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) h(K, k),$$

where

$$h(K,k) = \begin{cases} e^{-\Lambda T} \mathbb{E}\left[P_{BS}\left(S_0^{(k)}, \sigma, K\right)\right], & K \le K_k^*, \\ \left(1 - e^{-\Lambda T}\right)\left(K - \mathbb{E}\left[P_{BS}\left(S_0^{(k)}, \sigma, K\right)\right]\right) \\ \qquad - S_0^{(k)} e^{\mu T} \Phi(d(B,k)) - \sigma\sqrt{T}\Big), & else \end{cases}$$

for $K_k^* = S_0^{(k)} e^{d(B,k)\sigma\sqrt{T} + (\mu - 1/2\sigma^2)T}$ and $f(P_K) = \sqrt{e^{-\Lambda T}(1 - e^{-\Lambda T})(\mathbb{E}[P_K^2] - \mathbb{E}[P_K]^2)}$. Further, we can conclude that there is a unique $\widehat{K}^*$ maximizing the correlation above.

Example 5.4. **(Correlation example including derivative)** *The selected time horizon for this numerical analysis is* $T = 0.3$ *years, corresponding to approximately four months. Furthermore, the barrier B was estimated via the root search as depicted in* Figure 9. *We choose* $\theta_{asset} = 0.4$, *which means a jump of* N_T *results in a 40% decrease of* S_T. *Furthermore, Lemma* 5.2 *yields* $\lambda > \Lambda$, *restricting the choice of* λ. Figure 10 *displays the results:*

Additionally, Figure 11 displays the optimal quantity from Lemma 1.1 for the strike price established in Example 5.4.

6. Application to risk capital under solvency II

Building upon the foundation laid in the introduction, our goal is to ensure that

$$SCR_X - SCR_{X-P_K} > P_{K,0},$$

where P_K denotes the put option derived in preceding sections through the maximization of its correlation with the default indicator. In this section, we shift our focus to **S4** and demonstrate through numerical analysis that for each of the three models discussed, integrating the aforementioned put option into the portfolio significantly diminishes the risk capital, thereby mitigating the potential for unexpected loss. In the definition of the solvency capital requirement, α corresponds to 0.995 and represents the confidence level. Furthermore, in the parts of the analysis incorporating the option, the price for the option $P_{K,0}$ is always included. We note that, the addition of such put options should not result in a change in the expected loss but rather in the unexpected loss. This arises from the fact that while the payout of the put option is considered under $\mathbb{P}$, its price is evaluated under $\mathbb{Q}$ and is discounted over the time horizon T. Since the time horizon we consider is relatively short, the

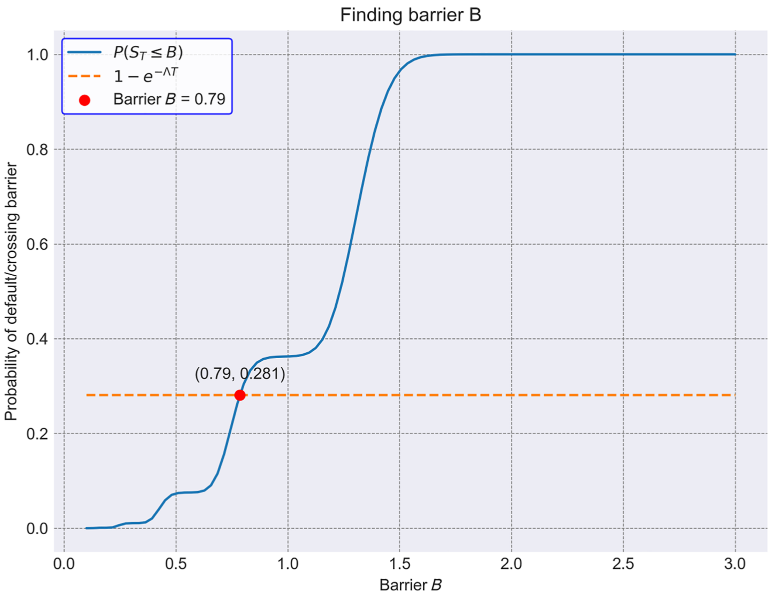

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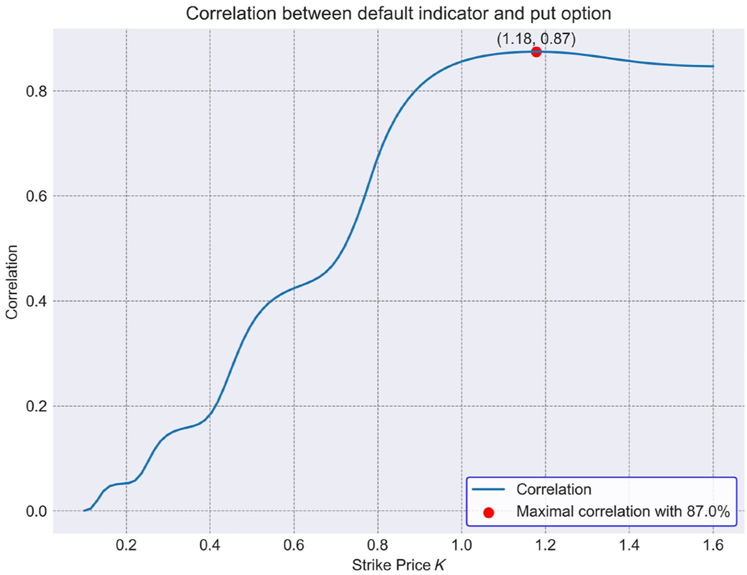
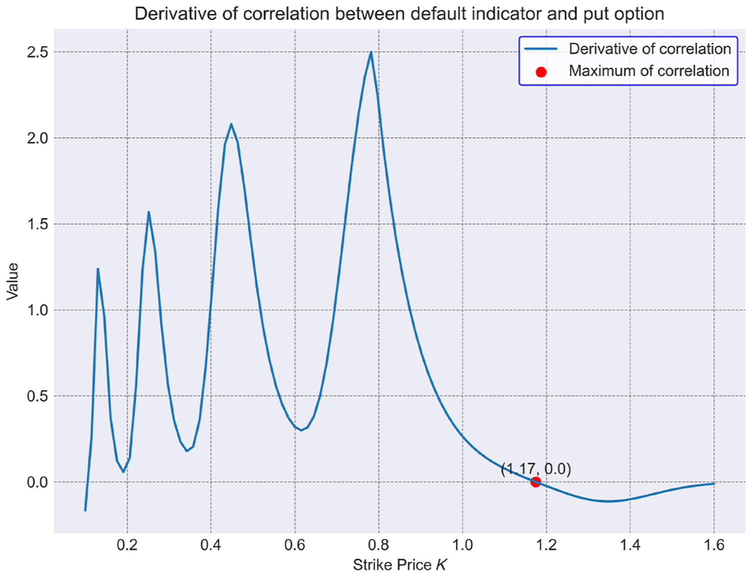
Figure 9. Numerical root search for B with parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\lambda = 1.5$, $\Lambda = 1.1$, $k = 5$



Correlation

(a)



Derivative of correlation

(b)

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Figure 10.
Numerical illustration of the correlation function and its derivative for the parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $\theta_{asset} = 0.4$

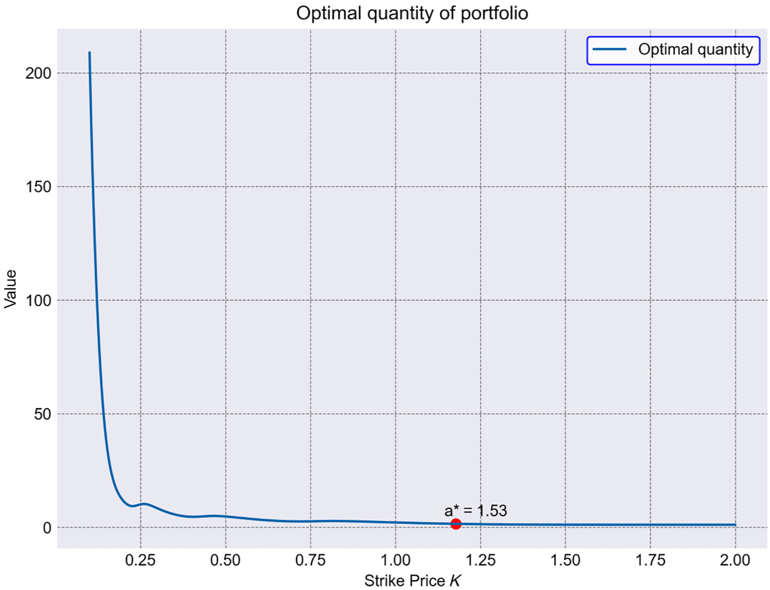

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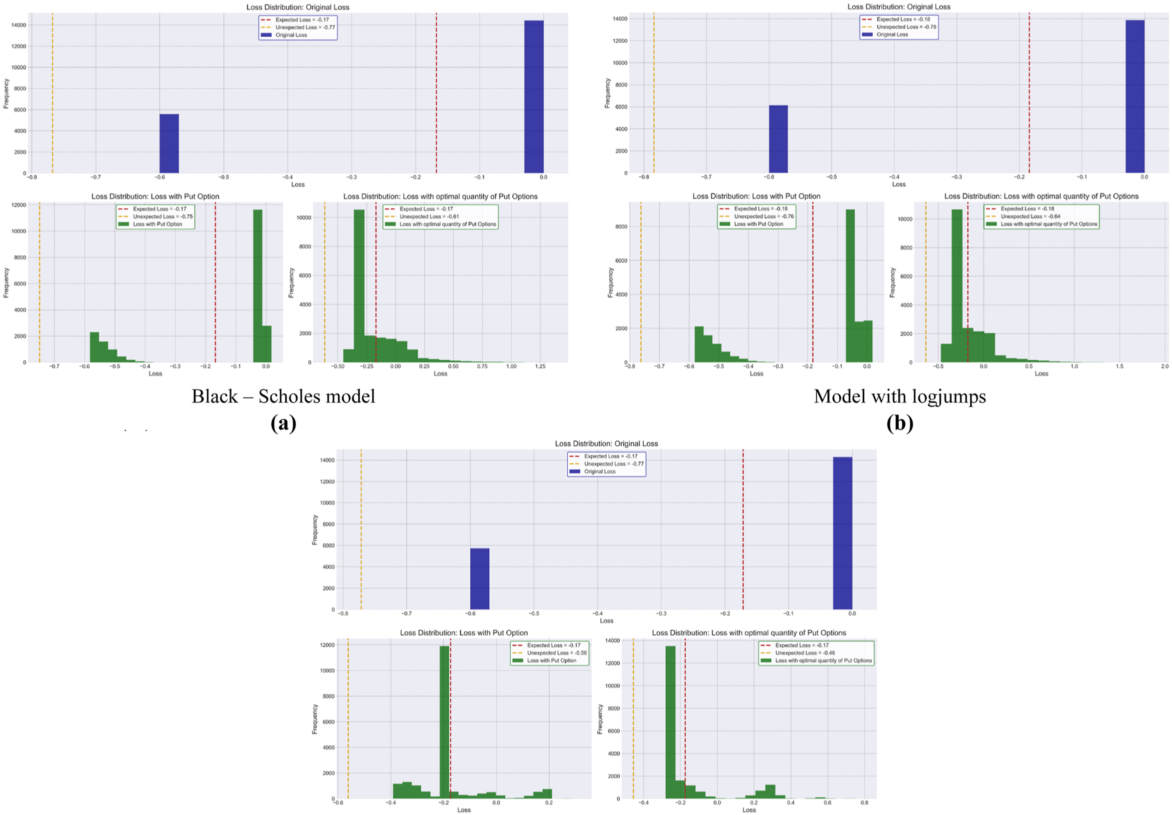
Figure 11. Numerical demonstration of optimal quantity utilizing parameters $S_0 = 1$, $\mu = 0.07$, $T = 0.3$, $\sigma = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $K^* = 1.18$, resulting in $a^* = 1.53$

difference arising from the change of measure should not be significantly large. Therefore, the shift in the expected loss should be minimal and is expected to be reflected in the subsequent numerical analysis. In our analysis across different markets, we strive for numerical consistency by adopting identical parameter values in the correlation function examples. It is important to note that the selection of $T = 0.3$ and $\Lambda = 1.1$ results in a medium default probability. This choice was deliberate, the rationale being that the impact of our analysis becomes increasingly pronounced with higher default probabilities. Therefore, we can validate our findings more robustly by focusing on the lower end of the default probability spectrum. This approach ensures that our conclusions are not merely artifacts of high-risk scenarios but are applicable across a broader range of conditions, reinforcing our results' validity. Figure 12 displays the numerical outcome for all considered markets.

In the respective upper graph, we observe the original loss distribution characterized by a binary profile, wherein losses are either absent or amount to $1 - \kappa = 0.6$, with $\kappa = 0.4$ representing the recovery value. Here, the exposure is defined as $C = 1$. In the expected lower graphs, we discern the altered loss profiles after the strategic incorporation of one put option into the portfolio on the left and the incorporation of a^* put options on the right. Notably, a shift of the unexpected loss towards the right is evident in all of the distributions. Adding the put option(s) has displaced the loss profile, signifying a notable alteration in the portfolio's risk dynamics. Nonetheless, as expected, the expected loss barely changed. Despite potential limitations in the underlying Black–Scholes model for S_T, the analysis of the loss profile aligns with our expectations. In comparison to the Black–Scholes model, our findings for the model with logjumps highlight a similar outcome. While the unexpected loss could be reduced, the expected loss remained similar, as expected. The final model we explored for S_T, inspired by the principles of Lévy factor models, incorporated a jump-diffusion model characterized by exclusively negative constant jumps. Compared to preceding models, the outcomes generated by this model appear exceptionally favorable. This observation is primarily attributed to the parameter selection strategy employed. As previously discussed,

Figure 12. Reduction in risk capital for the parameters $S_0 = 1$, $\hat{\mu} = 0.07$, $T = 0.3$, $\hat{\sigma} = 0.15$, $\Lambda = 1.1$, $\lambda = 1.5$, $\mu_J = 0.02$, $\sigma_J = 0.1$, $\mathbf{B} = \hat{\mathbf{B}} = (0.97, 0.97, 0.79)$, $\alpha = 0.995$ and the obtained strikes of the put option $\mathbf{K}^* = (1.02, 1.03, 1.18)$ (see Example 3.2, 4.2, 5.4)



Black – Scholes model
(a)

Model with logjumps
(b)

Model with constant negative jumps
(c)

Source(s): The figure is provided by the author we need to have $\Lambda < \lambda$ and $0 < \theta_{\text{asset}} < 1$. Hence, the modeler has two parameters to choose. Specifically, these parameters must be chosen so that the original loss is not significantly underestimated, which is challenging without comparisons to other models. Consequently, the choice of these parameters is crucial for the model's outcome, leading to an elevated model risk. Therefore, despite its intuitive appeal, this model should only be used for practical risk assessment tasks if the model risk can be adequately addressed. This concludes **S4**.

6.1 Comparison of numerical examples

In this section, we aimed to maintain consistency in our parameter choices to enable meaningful comparisons. We set the company's default intensity parameter at $\Lambda = 1.1$, corresponding to an approximate 28% default probability. The drift, volatility, and time were fixed at $(\mu, \sigma, T) = (0.07, 0.15, 0.3)$. The confidence level was inherently determined by SCR regulations, with $\alpha = 0.995$, and the initial value was set to $S_0 = 1$. For the jump-diffusion models, we selected Poisson process intensity parameters of $\lambda = 1.5$ for both the log-normal model and the model with constant negative jumps. Table 1 compares the outcomes of different parameters, specifically the obtained barrier for the given market parameters, the optimal strike with corresponding correlation and quantity, and the unexpected loss (UEL) both before and after hedging with a^* respective put options.

It becomes apparent that the model with constant negative jumps yields the smallest optimal quantity while still achieving the highest reduction in unexpected loss (UEL). As previously discussed, this outcome is significantly influenced by the choice of λ given the default intensity parameter Λ and θ_{asset}. In this numerical analysis, the parameters were selected so that the original loss closely matches that of the other two models, as reflected in the table. In practice, however, this comparison may not always be available, leaving the modeler with limited guidance for parameter choice beyond $\lambda > \Lambda$, thereby introducing elevated model risk. For this specific numerical example, the Black–Scholes model and the model with lognormal jumps exhibit similar behavior. Consequently, the choice rests between the simplicity of the Black–Scholes model and the more realistic framework of the jump-diffusion model with lognormal jumps. Further analysis could explore scenarios with varying jump size parameters.

7. Discussion and conclusion

This paper focused on mitigating existing credit debt through derivatives. This concept holds particular relevance for (insurance) companies who aim to retain the risk in-house while safeguarding against exposure. Diverging from conventional literature, our approach used structural default models assuming a pre-established default model. This strategic choice facilitates integrating this paper's insights into existing risk management frameworks. Furthermore, by adopting a realistic view that defaults can precede a

Parameter	Black–Scholes model	Lognormal jumps	Constant negative jumps
Barrier B	0.97	0.97	0.79
Correlation	*0.87*	0.86	*0.87*
Optimal Strike K^*	*1.02*	1.03	1.18
Optimal Quantity a^*	8.29	6.85	*1.53*
UEL	−0.77	−0.78	−0.77
UEL after hedging	−0.61	−0.64	−0.46
Difference in UEL	0.16	0.14	*0.31*

Source(s): The table is provided by the author

Table 1. Comparison of the models with italicized best values

complete depletion of company stock value, we transitioned from Merton's original asset value-based default distance to considering stock price breaches of a new, positive barrier $B > 0$. The loss in question was then defined relative to this stock price threshold in (1), including the maturity T. The objective was to bridge the previously established default model with any given model for S_T, imposing equivalence exclusively at maturity T. This equivalence at maturity T, crucial for comparing the loss profile against derivatives written on S_T, laid the groundwork for examining the efficacy of incorporating a European put option, denoted as P_K, in counterbalancing potential losses as S_T decreases. We realized that the introduced loss profile (1) closely resembles a digital put option as a function of S_T. We then further argued, as these derivatives are not traded, that one should be able to partially represent the loss function by a European put option P_K. As decreasing values in S_T lead to an elevated risk of a loss, the inclusion of such a put option, as it then increases in value, should be able to partially offset the resulting loss. Understanding that our goal was to achieve the best linear fit between the original loss X and the portfolio $a(P_K - P_{K,0})$, we defined Pearsons's correlation as the objective function in our optimization problem stated in (2). This approach ultimately determines a unique strike. Having established this strike, we addressed how many such options should be purchased in (3). We then commenced with the Black–Scholes model. As the most tractable model for S_T, it allowed for parameter estimation, as we were able to, in addition to the equivalence in T, estimate parameters based on a least squares criterion for distributions for all other time points. Next, we extended the model with lognormal jumps and observed similar behavior of the correlation function, as a unique maximum was found. Parameter estimation, however, is quite difficult as adding jumps renders the paths of S_t not continuous. Furthermore, the jump parameters extend the complexity. Hence, we imposed distributional equivalence in T and left the parameter estimation for further research. The third model we considered was motivated by the Lévy factor model introduced in Mai and Scherer (2017). Considering a jump-diffusion process with solely negative jumps, we also found a unique maximizer of the optimization problem. However, the constant jump size has to be chosen explicitly, reducing the applicability of this model. Comparative loss analyses across these models affirmed the anticipated outcomes, albeit with a critical view of the practical deployment of the constant negative jump model. In addition, adopting Pearson's correlation can also be challenged. Furthermore, alternate loss functions, increased model sophistication (e.g. via stochastic volatility), and refined parameter estimation for jump-diffusion processes could be considered. Enhancing the numerical analysis with different maturities T also allows future research. A significant aspect of our study is the focus on numerical analysis rather than empirical estimation, particularly in jump-diffusion models. While empirical estimation would involve parameter estimation from market data, especially for models incorporating jumps, this process is notoriously complex and falls outside the scope of our current work. Accurately capturing the dynamics of defaulted companies' stock data and corresponding historical put options poses considerable challenges, primarily due to the availability and quality of such data. Moreover, parameter estimation for jump models is intricate, requiring sophisticated techniques and assumptions to ensure reliability and accuracy. Although we have not undertaken empirical estimation in this study, future research could explore this avenue by developing robust methods for parameter estimation from market data. This would involve identifying suitable datasets and employing advanced statistical techniques to capture the nuances of market behavior in the presence of jumps. Such efforts would enhance the practical applicability of our approach and contribute to a deeper understanding of the empirical relationships between credit risk and equity markets. By addressing these challenges, future work can build on our findings and further refine the integration of credit risk models with real-world data. The optimization problem's underlying constraints, particularly concerning the SCR, might also warrant reevaluation. Moreover, while the methodology presupposes the availability of derivatives for listed, liquid companies, the idea can be extended to unlisted entities via correlated liquid indices, factoring in a "correlation error" to adjust the findings accordingly.

Notes

1. Merton originally introduced debt as the outstanding obligations with a fixed maturity T, and, hence, assumes debt to be constant over time $t \leq T$.
2. As the number of outstanding shares N is a constant, the default time is ultimately driven by the dynamics of S_t. This change from asset to stock price dynamics has also been used in Crosbie (2003) by scaling the volatility parameter accordingly; hence, this consideration is feasible.
3. The introduced threshold B is constant for the specific maturity T, and it is noteworthy that for small T, this barrier will be close to the current stock price. This proximity arises because the likelihood of the stock price falling substantially in a very short time interval is low, thus necessitating the default barrier to be relatively near the current price. This relationship is corroborated by the numerical analysis discussed later.
4. Given that the option's maturity aligns with that of the exposure and the model parameters are predetermined, the strike price becomes the sole variable.
5. The comparison of the to-be-paid price for the options implies a challenge in measuring the reduction in SCR in monetary units. This ambiguity complicates direct comparisons. However, this criterion sets the absolute minimum requirement that must be met. Further details on this topic will be explored in the numerical analysis of the SCR reduction later on.
6. This enables companies to adapt the results of this paper into their already existing risk management process.
7. This is done via the credit triangle, which approximates the hazard rate. For derivations, we refer the reader to [Packham *et al.*, 2013; Appendix].
8. Following the ISDA convention, the exponential law for the univariate default time of a company is a natural assumption.
9. We do this by finding a unique maximum of the objective function, first disregarding the capital constraint. This will be considered by numerical analysis in **S4**.
10. For a complete overview of default modeling and related topics within the structural approach, we refer to Acharya (2005), Lipton and Rennie (2011).
11. For mathematical compatibility we choose the maturity of the option to coincide with T, however, any available option at the market with maturity greater or equal than T will do. We propose choosing the first available maturity after T.
12. Given that in the Black–Scholes model, the drift rate is implicitly given as a function of the risk-less rate, the optimization could be reduced to two parameters. However, this implies knowledge about potential risk-less rates, and, hence, we include the parameter in the estimation as well.
13. This grid implies that there exists an $i \in [N]$ such that $t_i = T$.
14. In the above numerical example, we chose $T = 0.3$. We can obtain with the above formula, $B = 0.97$, which coincides with the estimation results, $\widehat{B}$.
15. The barrier in this particular example is relatively high, considering $S_0 = 1$. As argued before, this comes from the short time horizon $T = 0.3$, as the Brownian motion is unlikely to fall significantly in a short time window, and, hence, to reflect default, the barrier needs to be higher.
16. As the asset side assumes a relative change and the credit size a total change, it is natural to assume $\theta_{credit} \neq \theta_{asset}$.
17. Follows from the fact that the price of the put option increases in K.

References

Acharya, V. (2005), "Credit risk: pricing, measurement, and management", *Economica*, Vol. 72 No. 285, pp. 181-182, ISSN 0013-0427, doi: 10.1111/j.0013-0427.2005.408_3.x.

Aguilar, J.P., Pesci, N. and James, V. (2021), "A structural approach to default modelling with pure jump processes", *Applied Mathematical Finance*, Vol. 28 No. 1, pp. 48-78, doi: 10.1080/1350486X.2021.1957956.

Altman, E.I. (1968), "Financial ratios, discriminant analysis and the prediction of corporate bankruptcy", *The Journal of Finance*, Vol. 23 No. 4, pp. 589-609, doi: 10.1111/j.1540-6261.1968.tb00843.x.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999), "Coherent measures of risk", *Mathematical Finance*, Vol. 9 No. 3, pp. 203-228, ISSN 0960-1627, doi: 10.1111/1467-9965.00068.

Bai, J. and Wu, L. (2011), "Anchoring corporate credit swap spreads to firm fundamentals", *SSRN Electronic Journal*, ISSN 1556-5068, doi: 10.2139/ssrn.1787295, available at: http://www.ssrn.com/abstract=1787295

Bai, J., Goldstein, R.S. and Yang, F. (2020), "Is the credit spread puzzle a myth?", *Journal of Financial Economics*, Vol. 137 No. 2, pp. 297-319, ISSN 0304405X, doi: 10.1016/j.jfineco.2020.02.009, available at: https://linkinghub.elsevier.com/retrieve/pii/S0304405X20300428

Benos, A. and Papanastasopoulos, G. (2007), "Extending the Merton Model: a hybrid approach to assessing credit quality", *Mathematical and Computer Modelling*, Vol. 46 Nos 1-2, pp. 47-68, ISSN 08957177, doi: 10.1016/j.mcm.2006.12.012, available at: https://linkinghub.elsevier.com/retrieve/pii/S089571770700026X

Bernard, C., Moraux, F., Rüschendorf, L. and Vanduffel, S. (2015), "Optimal payoffs under state-dependent preferences", *Quantitative Finance*, Vol. 15, pp. 1157-1173, ISSN 1469-7688, doi: 10.1080/14697688.2014.981576.

Bharath, S.T. and Shumway, T. (2008), "Forecasting default with the Merton distance to default model", *Review of Financial Studies*, Vol. 21 No. 3, pp. 1339-1369, ISSN 0893-9454, doi: 10.1093/rfs/hhn044.

Black, F. and Cox, J.C. (1976), "Valuing corporate securities: some effects of bond indenture provisions", *The Journal of Finance*, Vol. 31 No. 2, pp. 351-367, ISSN 0022-1082, doi: 10.1111/j.1540-6261.1976.tb01891.x.

Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities", *Journal of Political Economy*, Vol. 81 No. 3, pp. 637-654, ISSN 0022-3808, doi: 10.1086/260062.

Brambilla, C., Gurny, M. and Lozza, S.O. (2015), "Structural credit risk models with Lévy processes: the VG and NIG cases", *Far East Journal of Mathematical Sciences (FJMS)*, Vol. 97 No. 1, pp. 101-119, ISSN 09720871, doi: 10.17654/FJMSMay2015_101_119, available at: http://www.pphmj.com/abstract/9081.htm

Carr, P. and Wu, L. (2011), "A simple robust link between American puts and credit protection", *Review of Financial Studies*, Vol. 24, pp. 473-505, ISSN 0893-9454, doi: 10.1093/rfs/hhq129.

Chen, W. and So, L. (2014), "Validation of the Merton distance to the default model under ambiguity", *Journal of Risk and Financial Management*, Vol. 7 No. 1, pp. 13-27, ISSN 1911-8074, doi: 10.3390/jrfm7010013, available at: http://www.mdpi.com/1911-8074/7/1/13

Cong, J., Tan, K.S. and Weng, C. (2013), "VAR-based optimal partial hedging", *ASTIN Bulletin*, Vol. 43 No. 3, pp. 271-299, ISSN 0515-0361, doi: 10.1017/asb.2013.19, available at: https://www.cambridge.org/core/product/identifier/S0515036113000196/type/journal_article

Cong, J., Tan, K.S. and Weng, C. (2014), "Conditional value-at-risk-based optimal partial hedging", *The Journal of Risk*, Vol. 16 No. 3, pp. 49-83, ISSN 14651211, doi: 10.21314/JOR.2014.293, available at: http://www.risk.net/journal-of-risk/technical-paper/2328154/conditional-value-at-risk-based-optimal-partial-hedging

Cont, R., Deguest, R. and Scandolo, G. (2010), "Robustness and sensitivity analysis of risk measurement procedures", *Quantitative Finance*, Vol. 10, pp. 593-606, ISSN 1469-7688, doi: 10.1080/14697681003685597.

Crosbie, P. (2003), *Modeling Default Risk - Modeling Methodology*, Moody's KMV, ISSN 15446123, available at: https://www.moodysanalytics.com/-/media/whitepaper/before-2011/12-18-03-modeling-default-risk.pdf

Cvitanić, J. and Spivak, G. (1999), "Maximizing the probability of a perfect hedge", *Annals of Applied Probability*, Vol. 9, ISSN 1050-5164, doi: 10.1214/aoap/1029962873, available at: https://projecteuclid.org/journals/annals-of-applied-probability/volume-9/issue-4/Maximizing-the-probability-of-a-perfect-hedge/10.1214/aoap/1029962873.full

Demchuk, A. and Gibson, R. (2006), "Stock market performance and the term structure of credit spreads", *Journal of Financial and Quantitative Analysis*, Vol. 41 No. 4, pp. 863-887, ISSN 0022-1090, doi: 10.1017/S0022109000002672, available at: https://www.cambridge.org/core/product/identifier/S0022109000002672/type/journal_article

Duffie, D. and Singleton, K.J. (1999), "Modeling term structures of defaultable bonds", *Review of Financial Studies*, Vol. 12 No. 4, pp. 687-720, ISSN 0893-9454, doi: 10.1093/rfs/12.4.687.

Duffie, D., Horel, G. and Saita, L. (2009), "Andreas Eckner, Guillaume Horel, and Leandro Saita. Frailty correlated default", *The Journal of Finance*, Vol. 64 No. 5, pp. 2089-2123, ISSN 0022-1082, doi: 10.1111/j.1540-6261.2009.01495.x.

Duyvesteyn, J. and Martens, M. (2015), "Forecasting sovereign default risk with Merton's model", *The Journal of Fixed Income*, Vol. 25 No. 2, pp. 58-71, ISSN 1059-8596, doi: 10.3905/jfi.2015.25.2.058.

Embrechts, P., Liu, H. and Wang, R. (2018), "Quantile-based risk sharing", *Operations Research*, Vol. 66 No. 4, pp. 936-949, doi: 10.1287/opre.2017.1716.

Föllmer, H. and Leukert, P. (1999), "Quantile hedging", available at: https://www.econstor.eu/bitstream/10419/61297/1/721138101.pdf

Föllmer, H. and Leukert, P. (2000), "Efficient hedging: cost versus shortfall risk", *Finance and Stochastics*, Vol. 4, pp. 117-146, ISSN 0949-2984, doi: 10.1007/s007800050008, available at:

Friewald, N., Wagner, C. and Zechner, J. (2014), "The cross-section of credit risk premia and equity returns", *The Journal of Finance*, Vol. 69 No. 6, pp. 2419-2469, doi: 10.1111/jofi.12143.

Geske, R. (1977), "The valuation of corporate liabilities as compound options", *The Journal of Financial and Quantitative Analysis*, Vol. 12 No. 4, p. 541, ISSN 00221090, doi: 10.2307/2330330, available at: https://www.jstor.org/stable/2330900?origin=crossref

Gordy, M.B. and Carey, M. (2007), "The Bank as grim reaper: debt composition and recoveries on defaulted debt", *SSRN Electronic Journal*, pp. 1556-5068, ISSN, doi: 10.2139/ssrn.972484, available at: http://www.ssrn.com/abstract=972484

Gray, D.F., Merton, R.C. and Bodie, Z. (2019), "Contingent claims approach to measuring and managing sovereign credit risk", *Journal of Investment Management*, Vol. 5, pp. 283-319, doi: 10.1142/9789814759618_0010.

Guo, N. and Li, L. (2022), "Stressed distance to default and default risk", *Journal of Credit Risk*, Vol. 18, ISSN 17559723, doi: 10.21314/JCR.2022.002, available at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3527349

Ibañez, A. (2023), "Default near-the-default-point: the value of and the distance to default", *SSRN Electronic Journal*, ISSN 1556-5068, doi: 10.2139/ssrn.4548010, available at: https://www.ssrn.com/abstract=4548010

Jessen, C. and Lando, D. (2015), "Robustness of distance-to-default", *Journal of Banking and Finance*, Vol. 50, pp. 493-505, ISSN 03784266, doi: 10.1016/j.jbankfin.2014.05.016, available at: https://linkinghub.elsevier.com/retrieve/pii/S0378426614001770

Jovan, M. and Ahčan, A. (2017), "Default prediction with the Merton-type structural model based on the NIG Lévy process", *Journal of Computational and Applied Mathematics*, Vol. 311, pp. 414-422, ISSN 03770427, doi: 10.1016/j.cam.2016.08.007, available at: https://linkinghub.elsevier.com/retrieve/pii/S0377042716303673

Leland, H.E. (1994), "Corporate debt value, bond covenants, and optimal capital structure", *The Journal of Finance*, Vol. 49 No. 4, pp. 1213-1252, ISSN 0022-1082, doi: 10.1111/j.1540-6261.1994.tb02452.x.

Leland, H.E. and Toft, K.B. (1996), "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads", *The Journal of Finance*, Vol. 51 No. 3, pp. 987-1019, ISSN 0022-1082, doi: 10.1111/j.1540-6261.1996.tb02714.x.

Lipton, A. and Rennie, A. (2011), *The Oxford Handbook of Credit Derivatives*, Oxford University Press, ISBN 0199546789, doi: 10.1093/oxfordhb/9780199546787.001.0001, available at: https://academic.oup.com/edited-volume/42049

Luciano, E. (2010), "Business time and new credit risk models", *SSRN Electronic Journal*, ISSN 1556-5068, doi: 10.2139/ssrn.1626726, available at: http://www.ssrn.com/abstract=1626726

Ma, Y. and Xu, W. (2016), "Structural credit risk modelling with Hawkes jump diffusion processes", *Journal of Computational and Applied Mathematics*, Vol. 303, pp. 69-80, ISSN 03770427, doi: 10.1016/j.cam.2016.02.032, available at: https://linkinghub.elsevier.com/retrieve/pii/S0377042716300796

Madan, D. and Schoutens, W. (2008), "Break on through to the single side", *The Journal of Credit Risk*, Vol. 4, pp. 3-20, ISSN 17446619, doi: 10.21314/JCR.2008.076, available at: http://www.risk.net/journal-of-credit-risk/technical-paper/2160726/break-single

Madan, D. and Schoutens, W. (2016), *Applied Conic Finance*, Cambridge University Press, ISBN 9781107151697, doi: 10.1017/CBO9781316585108, available at: https://www.cambridge.org/core/product/identifier/9781316585108/type/book

Mai, J.F. and Scherer, M. (2017), *Mai and Matthias Scherer. Simulating Copulas : Stochastic Models, Sampling Algorithms, and Applications*, World Scientific, available at: https://www.amazon.com/Simulating-Copulas-Stochastic-Applications-Quantitative/dp/9813149248

Marfe, R. (2012), "A multivariate pure-jump model with multi-factorial dependence structure", *International Journal of Theoretical and Applied Finance*, Vol. 15 No. 4, 1250028, ISSN 0219-0249, doi: 10.1142/S0219024912500288.

Melnikov, A. and Nosrati, A. (2017), *Equity-Linked Life Insurance*, Vol. 9, Chapman and Hall/CRC, ISBN 9781315118505, doi: 10.1201/9781315118505, available at: https://www.taylorfrancis.com/books/9781482240276

Melnikov, A. and Smirnov, I. (2012), "Dynamic hedging of conditional value-at-risk", *Insurance: Mathematics and Economics*, Vol. 51 No. 1, pp. 182-190, ISSN 01676687, doi: 10.1016/j.insmatheco.2012.03.011, available at: https://linkinghub.elsevier.com/retrieve/pii/S0167668712000467

Melnikov, A. and Wan, H. (2022), "On RVaR-based optimal partial hedging", *Annals of Actuarial Science*, Vol. 16 No. 2, pp. 349-366, ISSN 1748-4995, doi: 10.1017/S1748499521000269, available at: https://www.cambridge.org/core/product/identifier/S1748499521000269/type/journal_article

Merton, R.C. (1974), "On the pricing of corporate debt: the risk structure of interest rates", *The Journal of Finance*, Vol. 29 No. 2, p. 449, ISSN 00221082, doi: 10.2307/2978814, available at: https://www.jstor.org/stable/2978814?origin=crossref

Merton, R.C. (1976), "Option pricing when underlying stock returns are discontinuous", *Journal of Financial Economics*, Vol. 3 Nos 1-2, pp. 125-144, ISSN 0304405X, doi: 10.1016/0304-405X(76)90022-2, available at: https://linkinghub.elsevier.com/retrieve/pii/0304405X76900222

Michielon, M., Khedher, A. and Spreij, P. (2022), "Proxying credit curves via Wasserstein distances", *Annals of Operations Research*, Vol. 2 Nos 1-2, pp. 0254-5330, doi: 10.1007/s10479-022-04552-3.

Nakano, Y. (2011), *Partial Hedging for Defaultable Claims*, Springer, pp. 127-145, doi: 10.1007/978-4-431-53883-7_6.

Navas, J. (2003), "Correct calculation of volatility in a jump-diffusion model", *Journal of Derivatives*, Vol. 11 No. 2, pp. 66-72, doi: 10.3905/jod.2003.319217, available at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1031196

Nozawa, Y., Shi, Z. and Huang, J.-Z.J. (2019), "The global credit spread puzzle", *SSRN Electronic Journal*, ISSN 1556-5068, doi: 10.2139/ssrn.3453959, available at: https://www.ssrn.com/abstract=3453959

Packham, N., Schloegl, L. and Schmidt, W.M. (2013), "Credit gap risk in a first passage time model with jumps", *Quantitative Finance*, Vol. 13, pp. 1871-1889, ISSN 1469-7688, doi: 10.1080/14697688.2012.739729.

Pasricha, P., Lu, X. and Zhu, S.-P. (2021), "A note on the calculation of default probabilities in Structural credit risk modeling with Hawkes jump–diffusion processes", *Journal of Computational and Applied Mathematics*, Vol. 381, 113037, ISSN 03770427, doi: 10.1016/j.cam.2020.113037, available at: https://linkinghub.elsevier.com/retrieve/pii/S0377042720303289

Philip Jones, E., Mason, S.P. and Rosenfeld, E. (1984), "Contingent claims analysis of corporate capital structures: an empirical investigation", *The Journal of Finance*, Vol. 39 No. 3, pp. 611-625, ISSN 0022-1082, doi: 10.1111/j.1540-6261.1984.tb03649.x.

Rudin, W. (1976), *Principles of Mathematical Analysis*, International series in pure and applied mathematics. McGraw-Hill, ISBN 9780070856134, available at: https://books.google.de/books?id=kwqzPAAACAAJ

Siggelkow, C. and Fernandez, R.M. (2023), "SME default prediction using random forest including nonfinancial features: an empiricial analysis of German enterprises", *Journal of the International Council for Small Business*, Vol. 5 No. 2, pp. 1-19, ISSN 2643-7015, doi: 10.1080/26437015.2023.2224108.

Tankov, P. (2003), *Financial Modelling with Jump Processes*, Vol. 12, Chapman and Hall/CRC, ISBN 9781135437947, doi: 10.1201/9780203485217, available at: https://www.taylorfrancis.com/books/9781135437947

Vassalou, M. and Xing, Y. (2004), "Default risk in equity returns", *The Journal of Finance*, Vol. 59 No. 2, pp. 831-868, ISSN 0022-1082, doi: 10.1111/j.1540-6261.2004.00650.x.

White, R. (2014), "The pricing and risk management of credit default swaps , with a focus on the ISDA model", *OpenGamma Quantitative Research*, Vol. 16, available at: https://quant.opengamma.io/Pricing-and-Risk-Management-of-Credit-Default-Swaps-OpenGamma.pdf

Xu, M. (2006), "Risk measure pricing and hedging in incomplete markets", *Annals of Finance*, Vol. 2, pp. 51-71, ISSN 1614-2446, doi: 10.1007/s10436-005-0023-x.

Zhou, C. (1997), "A jump-diffusion approach to modeling credit risk and valuing defaultable securities", *Finance and Economics Discussion Series*, pp. 1-47, 5 1997. ISSN 1936-2854, doi: 10.17016/feds.1997.15, available at: https://www.federalreserve.gov/econres/feds/a-jump-diffusion-approach-to-modeling-credit-risk-and-valuing-defaultable-securities.htm

Appendix

Proofs

Proof of Theorem 3.1. The proof consists of three main steps, i.e.

(i) Derive the correlation function in this model by calculating each quantity.

(ii) Derive the derivative of the correlation function by calculating the derivative of each part and piecing them together.

(iii) Show that there exists a K^* such that

$$\forall K < K^* : \frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) > 0,$$

$$\forall K > K^* : \frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) < 0.$$

This yields the existence of a maximum. For uniqueness, we then show

$$\lim_{K\to\infty} \frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) = 0.$$

Step (i): Derivation of the correlation function

Expectation and variance of loss indicator

With the connection to the existing default model, we can see that

$$\mathrm{Var}\left(\mathbb{1}_{\{S_T\le B\}}\right)\overset{(7)}{=}\left(1-e^{-\Lambda T}\right)-\left(1-e^{-\Lambda T}\right)^2=e^{-\Lambda T}\left(1-e^{-\Lambda T}\right).$$

Furthermore, we have

$$\mathbb{E}\left[\mathbb{1}_{\{S_T\le B\}}\right]=\mathbb{P}(S_T\le B)=^{(7)}1-e^{-\Lambda T}.$$

Expectation and variance of put option payout

We know

$$\mathbb{E}[P_K]=K\Phi(d(K))-S_0e^{\mu T}\Phi\left(d(K)-\sigma\sqrt{T}\right), \tag{A.1}$$

where

$$d(K)=\frac{\ln\left(\frac{K}{S_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \tag{A.2}$$

Next, we have a look at the second moment, i.e.

$$\begin{aligned}\mathbb{E}\left[P_K^2\right]&=\mathbb{E}\left[\mathbb{1}_{\{K>S_T\}}\left(K-S_0e^{\left(\mu-1/2\sigma^2\right)T+\sigma W_T}\right)^2\right]\\&=K^2\Phi(d(K))-2KS_0e^{\mu T}\Phi\left(d(K)-\sigma\sqrt{T}\right)\\&\quad+S_0^2e^{2\left(\mu+1/2\sigma^2\right)T}\Phi\left(d(K)-2\sigma\sqrt{T}\right),\end{aligned} \tag{A.3}$$

where in the last step, we have used a quadratic expansion again. With these results, we also obtain the variance of P_K.

Covariance between loss indicator and put option

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\{S_T\le B\}}P_K\right]&=\mathbb{E}\left[\mathbb{1}_{\left\{\frac{1}{\sqrt{T}}W_T\le d(B)\right\}}\mathbb{1}_{\{K>S_T\}}\left(K-S_0e^{\left(\mu-1/2\sigma^2\right)T+\sigma W_T}\right)\right]\\&=^{(A1)}K\Phi(\min\{d(B),d(K)\})-S_0e^{\mu T}\Phi\left(\min\{d(B),d(K)\}-\sigma\sqrt{T}\right).\end{aligned}$$

Using this, we calculate the covariance via:

$$\mathrm{Cov}\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) = \mathbb{E}\left[\mathbb{1}_{\{S_T \le B\}} P_K\right] - \mathbb{E}\left[\mathbb{1}_{\{S_T \le B\}}\right]\mathbb{E}[P_K]$$
$$= \begin{cases} e^{-\Lambda T}\mathbb{E}[P_K] & \text{for } K \le B, \\ \left(1 - e^{-\Lambda T}\right)(K - \mathbb{E}[P_K]) - S_0 e^{\mu T}\Phi(d(B)) - \sigma\sqrt{T}\Big) & \text{for } K > B, \end{cases} \quad \text{(A.4)}$$

where B is taken from (7). Putting (A.1), (A.3) and (A.4) together yields the correlation function. Furthermore, we can rearrange (7) to find $d(B) = \Phi^{-1}(1 - e^{-\Lambda T})$ explicitly in this model.

Step (ii): Derivation of the derivative

Next, we calculate the derivative for each part and, ultimately, the correlation function.

Derivative of moments of put option payout

We start with the price of the put:

$$\frac{\partial \mathbb{E}[P_K]}{\partial K} = \Phi(d(K)) + \phi(d(K))\frac{1}{\sigma\sqrt{T}} - S_0 e^{\mu T}\phi\left(d(K) - \sigma\sqrt{T}\right)\frac{1}{\sigma\sqrt{T}K}.$$

Next, we calculate the derivative of the second moment of P_K:

$$\frac{\partial \mathbb{E}\left[P_K^2\right]}{\partial K} \overset{\text{(A3)}}{=} = 2K\Phi(d(K)) + K\phi(d(K))\frac{1}{\sigma\sqrt{T}} - 2S_0 e^{\mu T}\Phi\left(d(K) - \sigma\sqrt{T}\right)$$
$$- 2S_0 e^{\mu T}\phi\left(d(K) - \sigma\sqrt{T}\right)\frac{1}{\sigma\sqrt{T}} + S_0^2 e^{2\left(\mu + 1/2\sigma^2\right)T}\phi\Big(d(K)$$
$$- 2\sigma\sqrt{T}\Big)\frac{1}{\sigma\sqrt{T}K}.$$

Derivative of covariance of loss indicator and put option payout

Using this result, we can calculate:

$$\frac{\partial}{\partial K}\mathrm{Cov}\left(\mathbb{1}_{\{S_T \le B\}}, P_K\right) = \begin{cases} e^{-\Lambda T}\dfrac{\partial \mathbb{E}[P_K]}{\partial K} & \text{for } K < B, \\ \left(1 - e^{-\Lambda T}\right)\left(1 - \dfrac{\partial \mathbb{E}[P_K]}{\partial K}\right) & \text{for } K > B. \end{cases}$$

Before continuing further, we owe the reader an argument for differentiability in $K = B$. Hence, we check if the left limit agrees with the right limit. With the continuity of the derivative of the price of the put, we then have the following condition:

$$\lim_{K \nearrow B} e^{-\Lambda T} \frac{\partial \mathbb{E}[P_K]}{\partial K} = \lim_{K \searrow B} 1 - e^{-\Lambda T} - \frac{\partial \mathbb{E}[P_K]}{\partial K} + e^{-\Lambda T} \frac{\partial \mathbb{E}[P_K]}{\partial K}$$
$$\Leftrightarrow \lim_{K \searrow B} \frac{\partial \mathbb{E}[P_K]}{\partial K} = 1 - e^{-\Lambda T}. \tag{A.5}$$

Now, we calculate with the definition of $d(K)$:

$$\begin{aligned}\lim_{K \searrow B} \frac{\partial \mathbb{E}[P_K]}{\partial K} &\overset{(7)}{=} 1 - e^{-\Lambda T} + \phi(d(B)) \frac{1}{\sigma\sqrt{T}} - S_0 e^{\mu T} \phi\left(d(B) - \sigma\sqrt{T}\right) \frac{1}{\sigma\sqrt{T}B} \\ &= 1 - e^{-\Lambda T} + \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} e^{-d(B)^2/2} \left(1 - \frac{S_0 e^{\mu T}}{B} e^{\ln(B/S_0) - \mu T}\right) = 1 - e^{-\Lambda T},\end{aligned}$$

completing (A.5), and, hence, we write $K \leq B$ in the derivative of the covariance above.

Derivative of $1/f(P_K)$

Using the chain and product rule, we obtain:

$$\begin{aligned}\frac{\partial}{\partial K} \frac{1}{f(P_K)} &= -\frac{1}{2}\left(e^{-\Lambda T}\left(1 - e^{-\Lambda T}\right) \mathrm{Var}(P_K)\right)^{-\frac{3}{2}} \\ &\times e^{-\Lambda T}\left(1 - e^{-\Lambda T}\right)\left(\frac{\partial}{\partial K} \mathrm{Var}(P_K)\right),\end{aligned} \tag{A.6}$$

where $f(P_K) = \sqrt{e^{-\Lambda T}(1 - e^{-\Lambda T})(\mathrm{Var}(P_K))}$.

Derivative of correlation function

In total, we then get

$$\begin{aligned}\frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right) &= \frac{\partial}{\partial K} \mathrm{Cov}\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right) \frac{1}{f(P_K)} + \mathrm{Cov}\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right) \frac{\partial}{\partial K} \frac{1}{f(P_K)} \\ &= \begin{cases} \dfrac{e^{-\Lambda T}}{f(P_K)} \dfrac{\partial \mathbb{E}[P_K]}{\partial K} + e^{-\Lambda T} \mathbb{E}[P_K] \dfrac{\partial}{\partial K} \dfrac{1}{f(P_K)} & \text{for } K \leq B, \\ \dfrac{1 - e^{-\Lambda T}}{f(P_K)} \left(1 - \dfrac{\partial \mathbb{E}[P_K]}{\partial K}\right) & \\ + \left(\left(1 - e^{-\Lambda T}\right)(K - \mathbb{E}[P_K]) - A\right) \dfrac{\partial}{\partial K} \dfrac{1}{f(P_K)} & \text{for } K > B, \end{cases}\end{aligned} \tag{A.7}$$

where $A = S_0 e^{\mu T} \Phi\left(d(B) - \sigma\sqrt{T}\right)$.

Step (iii): Existence and uniqueness

In general, we know that the variance of the put option decreases with increasing K, as higher strike prices lead to lower sensitivity of the option price to the underlying stock price, thus reducing the variance. Hence, using the definition of $f(P_K)$ and (A.6), we have

$$\frac{1}{f(P_K)} \geq 0, \quad \frac{\partial}{\partial K}\frac{1}{f(P_K)} \geq 0. \tag{A.8}$$

Therefore, we see that the correlation function is increasing for all $K \leq B$ [17] and, hence, the derivative is positive for this part. Therefore, any extremum of this function should be larger than B, and hence, we focus on the right side of the correlation function and its derivative.

Existence

For all $K > B$, we reformulate the derivative via

$$\frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right) = \left(1 - e^{-\Lambda T}\right)\frac{\partial}{\partial K}\left(\frac{1}{f(P_K)}(K - \mathbb{E}[P_K])\right) - \frac{\partial}{\partial K}\left(\frac{1}{f(P_K)}A\right).$$

Let $g(K) := K - \mathbb{E}[P_K]$. We clearly know that $g(K) \geq 0$, $g(K)$ is monotonically decreasing, and, hence,

$$\frac{\partial}{\partial K} g(K) < 0. \tag{A.9}$$

Lastly, as $\mathbb{E}[P_K]$ approaches K for large strikes, we have

$$\lim_{K \to \infty} g(K) = 0. \tag{A.10}$$

Putting these arguments together, we can see that

$$\exists K^* : \forall K > K^* : g(K) < C := \frac{A}{1 - e^{-\Lambda T}}. \tag{A.11}$$

But then we have that there exists a K^* such that for all $K > K^*$, we have

$$\begin{aligned}
&\frac{\partial}{\partial K} Corr\left(\mathbb{1}_{\{S_T \leq B\}}, P_K\right) \\
&= \left(1 - e^{-\Lambda T}\right)\left(\left(\frac{\partial}{\partial K}\frac{1}{f(P_K)}\right)g(K) + \left(\frac{\partial}{\partial K}g(K)\right)\frac{1}{f(P_K)}\right) - \frac{\partial}{\partial K}\left(\frac{1}{f(P_K)}A\right) \\
&\overset{(A.9),(A.8)}{<} \left(1 - e^{-\Lambda T}\right)\left(\frac{\partial}{\partial K}\frac{1}{f(P_K)}\right)g(K) - \frac{\partial}{\partial K}\left(\frac{1}{f(P_K)}A\right) \\
&\overset{(A.11)}{<} \left(1 - e^{-\Lambda T}\right)\left(\frac{\partial}{\partial K}\frac{1}{f(P_K)}\right)\frac{A}{1 - e^{-\Lambda T}} - \frac{\partial}{\partial K}\left(\frac{1}{f(P_K)}A\right) = 0.
\end{aligned}$$

This means, that there exists K^* such that for all $K < K^*$ the derivative of the correlation is positive and for all $K > K^*$ negative, and, hence, a maximum.

Uniqueness

Uniqueness follows if we show that for all $K > K^*$, the derivative of the correlation converges to 0. First, due to (A.10), $g(K)$ monotonically decreasing and non-negative, and

$$\lim_{K\to\infty}\frac{1}{f(P_K)} = \frac{1}{\sqrt{e^{-\Lambda T}\left(1-e^{-\Lambda T}\right)\mathrm{Var}(S_T)}} = \frac{1}{\sqrt{e^{-\Lambda T}\left(1-e^{-\Lambda T}\right)S_0^2 e^{2\mu T}\left(e^{\sigma^2 T}-1\right)}},$$

we see from (A.7), it suffices to have

$$\lim_{K\to\infty}\frac{\partial}{\partial K}\frac{1}{f(P_K)} = 0.$$

However, as we have

$$\lim_{K\to\infty}\frac{\partial}{\partial K}\frac{1}{f(P_K)}$$

$$= -\frac{1}{2}\left(e^{-\Lambda T}\left(1-e^{-\Lambda T}\right)S_0^2 e^{2\mu T}\left(e^{\sigma^2 T}-1\right)\right)^{-\frac{3}{2}} e^{-\Lambda T}\left(1-e^{-\Lambda T}\right)\lim_{K\to\infty}\left(\frac{\partial}{\partial K}\mathrm{Var}(P_K)\right),$$

this follows from Lemma A.1.

Lemma A.1. **(Limit of derivative of variance of put option)** *We have*

$$\lim_{K\to\infty}\left(\frac{\partial}{\partial K}\mathrm{Var}(P_K)\right) = 0.$$

Proof. First, since $h(K) := \mathrm{Var}(P_K)$ is monotonically decreasing and converging to $L = \mathrm{Var}(S_T) = S_0^2 e^{2\mu T}\left(e^{\sigma^2 T}-1\right)$, we have

$$\forall \varepsilon > 0\ \exists \delta > 0 : |h(K) - L| < \varepsilon\ \forall K > \delta. \tag{A.12}$$

As $h(K)$ is also differentiable, we can apply the Mean-Value-Theorem (see, e.g. [Rudin, 1976, Theorem 5.10.]) to see that for any interval $[K, K+1]$ for large $K > \delta$, there exists $K^* \in (K, K+1)$ such that

$$\frac{\partial}{\partial K}h\left(K^*\right) = h(K+1) - h(K).$$

With this, we can use the definition of the limit again:

$$\left|\frac{\partial}{\partial K}h\left(K^*\right) - 0\right| = |h(K+1) - h(K)| \leq |h(K+1) - L| + |h(K) - L| \stackrel{(A12)}{=} 2\varepsilon \ \forall K^* > K > \delta.$$

As $K \to \infty$ implies $K^* \to \infty$ by construction ($K^* > K > \delta$), we have

$$\lim_{K\to\infty}\frac{\partial}{\partial K}h(K) = \lim_{K\to\infty}\frac{\partial}{\partial K}h\left(K^*\right) = 0.$$

Proof of Theorem 4.1. As, conditioned on the number of jumps, this model relates to the Black–Scholes model, we aim at applying the arguments from the proof of Theorem 3.1, albeit different parameters, namely different starting value, adjusted volatility, and adjusted (A.2). To apply all arguments, especially those including limits, we need to ensure exchangeability of the limit w.r.t. the strike and the infinite sum stemming from the number of jumps. For this, we consider

$$\sum_{k=0}^{\infty} g(K,k) := \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)h(K,k).$$

Here, $h(K, k)$ relates to the Black–Scholes model for any fixed k, and, hence, following the arguments in the proof of Theorem 3.1, we have

$$\exists M > 0 : |h(K,k)| \leq M.$$

Furthermore, we know that the density of a compound Poisson distribution decays exponentially fast, and, hence, we have

$$\exists C > 0, \alpha > 0 : |P(N_T = k)| \leq Ce^{-\alpha k}.$$

This, however, implies that an integrable function dominates $g(K, k)$, and, hence, the dominated convergence theorem yields

$$\lim_{K\to\infty}\sum_{k=0}^{\infty}\mathbb{P}(N_T = k)h(K,k) = \sum_{k=0}^{\infty}\mathbb{P}(N_T = k)\lim_{K\to\infty}h(K,k)\,.$$

With this, all the arguments from Theorem 3.1 can be used with the adjusted parameters stated above. We note that the constant limit for the variance differs, as can be found in Navas (2003). Additionally, in contrast to the Black–Scholes model, we cannot derive $d(B, k)$ explicitly but only by root search (see, Figure (6)).

Proof of Lemma 5.1. Following the arguments in the proof of Merton (1976), we aim to solely focus on pure jump effects, and, hence, we must eliminate the incremental drift resulting from the jumps. Given that these are relative changes, we have $\mathbb{E}[1 - \theta_{asset}] = 1 - \theta_{asset}$. As the jumps are solely negative, we obtain the following final form of the SDE:

$$\frac{dS_t}{S_t} = (\mu + \lambda(1 - \theta_{asset}))dt + \sigma dW_t + (1 - \theta_{asset})dN_t,$$

We then deploy [Tankov, 2003, Proposition 8.14] to find the solution.

Proof of Lemma 5.2. As we impose distributional equivalence in T, we need to ensure (11). Furthermore, we also describe default by checking if, at time T, the Lévy subordinator $\theta_{credit} N_T$ has crossed the threshold $\varepsilon \sim \exp(1)$. Hence, we need to choose the jump size, such that we ensure

$$\mathbb{P}(\theta_{credit} N_T \geq \varepsilon)(11) = 1 - e^{-\Lambda T}.$$

In particular, we need to have

$$\begin{aligned} \mathbb{P}(\theta_{credit} N_T \geq \varepsilon) &= \sum_{k=0}^{\infty} \mathbb{P}(N_T = k)\left(1 - e^{-\theta_{credit} k}\right) = 1 - e^{-\lambda T} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{k!} e^{-\theta_{credit} k} \\ &= 1 - e^{-\lambda T\left(1 - e^{-\theta_{credit}}\right)} \stackrel{!}{=} 1 - e^{-\Lambda T}. \end{aligned}$$

The claim follows.

Proof of Theorem 5.3. See proof of Theorem 4.1. The only difference is the parameters (no adjusted volatility, slightly different starting value) and, therefore, the constant limit of the variance of S_T.

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