A generalization of Ascoli–Arzelá theorem in C^n with application in the existence of a solution for a class of higher-order boundary value problem

Salah Benhiouna
Department of Mathematics, Universite Badji Mokhtar Annaba, Annaba, Algeria
Azzeddine Bellour
Laboratoire de Mathématiques appliquées et didactique, Ecole Normale Supérieure de Constantine, Constantine, Algeria, and
Rachida Amiar
Department of Mathematics, Universite Badji Mokhtar Annaba, Annaba, Algeria

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Abstract

Purpose – A generalization of Ascoli–Arzelá theorem in Banach spaces is established. Schauder's fixed point theorem is used to prove the existence of a solution for a boundary value problem of higher order. The authors' results are obtained under, rather, general assumptions.

Design/methodology/approach – First, a generalization of Ascoli–Arzelá theorem in Banach spaces in C^n is established. Second, this new generalization with Schauder's fixed point theorem to prove the existence of a solution for a boundary value problem of higher order is used. Finally, an illustrated example is given.

Findings – There is no funding.

Originality/value – In this work, a new generalization of Ascoli–Arzelá theorem in Banach spaces in C^n is established. To the best of the authors' knowledge, Ascoli–Arzelá theorem is given only in Banach spaces of continuous functions. In the second part, this new generalization with Schauder's fixed point theorem is used to prove the existence of a solution for a boundary value problem of higher order, where the derivatives appear in the non-linear terms.

Keywords Generalization of Ascoli–Arzelá theorem, Higher-order boundary value problem, Fixed point theorem

Paper type Research paper

1. Introduction

In this paper, we consider the following higher-order boundary value problem:

$$\begin{cases} u^{(n)} + f\left(t, u, u', \ldots, u^{(n-2)}\right) = 0, n \geq 2, t \in I = [0, 1], \\ u^{(i)}(0) = 0, 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \tag{1.1}$$

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where n is a given positive integer, $\alpha,\ \gamma\ >\ 0,\ \beta,\ \delta\ \geq\ 0$; f is continuous and satisfies $|f(s,u_0,u_1,\ldots,u_{n-2})|\leq a(s)+\sum_{k=0}^{n-2}b_k|u_k|$ such that a is continuous on I and $b_k\in\mathbb{R}^+, k=0,\ldots,n-2$.

Equation (1.1) and its particular forms have been studied by many authors (see, for example, [1–4, 6, 7, 9–13] and the references therein).

Wong and Agarwal in [13] and Patricia *et al.* in [12] have studied the following boundary value problem:

$$\begin{cases} u^{(n)}+\lambda Q\big(t,u,u',\ldots,u^{(n-2)}\big)=\lambda P\big(t,u,u',\ldots,u^{(n-2)}\big)\\ u^{(i)}(0)=0, 0\leq i\leq n-3,\\ \alpha u^{(n-2)}(0)-\beta u^{(n-1)}(0)=0,\\ \gamma u^{(n-2)}(1)+\delta u^{(n-1)}(1)=0,\end{cases}$$

under the following condition: there exists continuous functions f: $(0,+\infty)\rightarrow(0,+\infty)$ and $p_1,p,q_1,q:(0,1)\rightarrow\mathbb{R}$ such that

$$(i)\ q(t)\leq\frac{Q(t,u_0,u_1,\ldots,u_{n-2})}{f(u)}\leq q_1(t), p(t)\leq\frac{P(t,u_0,u_1,\ldots,u_{n-2})}{f(u)}\leq p_1(t).$$

$$(ii)\ q(t)-p_1(t)\geq 0.$$

Agarwal and Wong [1] have studied the existence of a positive solution for the problem (1.1) under the following condition: there exists $L\geq 0$ such that

$$f\big(t,u,u',\ldots,u^{(n-2)}\big)+L\geq 0 \text{ on } [0,1]\times[0,\infty)^{n-1},$$

$$\int_0^1 g(s,s)\big[f\big(s,u,u',\ldots,u^{(n-2)}\big)+L\big]ds\leq\lambda,$$

and some other conditions, where the function g is defined in (3.2).

Chyan and Henderson [3] have studied the existence of a positive solution of the following problem:

$$\begin{cases} u^{(n)}+\lambda q(t)f(u)=0,\\ u^{(i)}(0)=u^{(n-2)}(1)=0, 0\leq i\leq n-2.\end{cases}$$

such that f and q are continuous and non-negative functions.

The following analogical problem has been studied by Eloe and Ahmad in [5],

$$\begin{cases} u^{(n)}+f(t,u)=0, t\in(0,1)\\ u^{(i)}(0)=0, 0\leq i\leq n-2,\\ \alpha u(\eta)=u(1), 0<\eta<1,\end{cases}$$

The following more general form has been studied by J. R. Graef and T. Moussaoui in [8],

$$\begin{cases} u^{(n)}+f(t,u)=0, t\in(0,1)\\ u^{(i)}(0)=0, 0\leq i\leq n-2,\\ \sum_{i=1}^{m-2}\alpha_i u(\eta_i)=u(1), 0<\eta<1,\end{cases}$$

where the derivatives $x^{(i)}$, $0\leq i\leq n-2$ do not appear in the non-linear terms.

Our main task in this paper consists of giving a generalization of Ascoli–Arzelá theorem in the space $C^n(X,E)$ (the space of functions from a compact subset of $\mathbb{R}$ into a Banach space E

with continuous nth derivative) in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^n to prove the existences of a solution for the higher-order boundary value problem (1.1).

The rest of this paper is organized as follows. In Section 2, we give a generalization of Ascoli–Arzelá theorem in the space C^n. The existences of a solution to higher-order boundary value problem (1.1) are presented in Section 3.

2. A generalization of Ascoli–Arzelá theorem in C^n

Before stating the main result in this section, we provide the following notations and definition:

Let E be a Banach space endowed with the norm $\|.\|_1$, and X be a compact subset of $\mathbb{R}$. We note by $C^n(X, E)$ the space of all functions with n continuous derivatives from X to E; this space is endowed with the norm $\|f\| = \sum_{i=0}^{n} \|f^{(i)}\|_\infty$ such that $\|f\|_\infty = \sup_{x\in X}\{\|f(x)\|_1\}$.

For our purpose, we need the following definition in $C^n(X, E)$.

Definition 2.1. *The family $F \subset C^n(X, E)$ is called equi-continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that $\left\|f^{(i)}(x) - f^{(i)}(y)\right\|_1 < \varepsilon$ for all $i = 0, \ldots, n$ and for all $x, y \in X$ satisfying $|x - y| < \delta$.*

The family $F \subset C^n(X, E)$ is called equi-bounded if there is a constant M such that $\left\|f^{(i)}(x)\right\|_1 \leq M$ for all $i = 1, \ldots, n$, for all $f \in F$ and for all $x \in X$.

The following result gives the Ascoli–Arzelá theorem in the space $C^n(X, E)$

Theorem 2.2. *Let F be a subset of $C^n(X, E)$. Then F is relatively compact if and only if F is equi-continuous and equi-bounded.*

Proof. Assume that F is relatively compact. This means that $\overline{F}$ is compact. We claim that F is equi-continuous and equi-bounded. Since $\overline{F}$ is compact, then it is equi-bounded and since $F \subset \overline{F}$, we deduce that F is equi-bounded.

To see that F is equi-continuous, let $\varepsilon > 0$, then there exists $f_1, \ldots, f_m \in C^n(X, E)$ such that

$$F \subset B_{\frac{\varepsilon}{3(n+1)}}(f_1) \cup \ldots \cup B_{\frac{\varepsilon}{3(n+1)}}(f_m).$$

Since $f_j^{(i)}$ are uniformly continuous, then there exists $\delta > 0$ such that for all $x, y \in X$, if $|x - y| < \delta$, then for all $i = 0, \ldots, n$ and for all $j = 1, \ldots, m$

$$\|f_j^{(i)}(x) - f_j^{(i)}(y)\|_1 < \frac{\varepsilon}{3}.$$

Let $f \in F$, then there exists $j \in \{1, \ldots, m\}$ such that $f \in B_{\frac{\varepsilon}{3}}(f_j)$.

Hence, for all $i = 0, \ldots, n$

$$\|f^{(i)}(x) - f^{(i)}(y)\|_1 \leq \|f^{(i)}(x) - f_j^{(i)}(x)\|_1 + \left\|f_j^{(i)}(x) - f_j^{(i)}(y)\right\|_1 + \left\|f^{(i)}(y) - f_j^{(i)}(y)\right\|_1 < \varepsilon.$$

which implies that F is equi-continuous.

Conversely, assume that F is equi-continuous and equi-bounded. To show that F is relatively compact it suffices to show that F is totally bounded; indeed if F is totally bounded, then $\overline{F}$ is also totally bounded, which implies that $\overline{F}$ is compact.

Since F is equi-continuous, then for all $x \in X$ and $\varepsilon > 0$, there exists $\delta_x > 0$ such that if $y \in X$ and $|x - y| < \delta_x$, we have for all $i = 0, \ldots, n$

$$\|f^{(i)}(x) - f^{(i)}(y)\|_1 < \frac{\varepsilon}{4(n+1)} \text{ for all } f \in F.$$

The collection $\{B_{\delta_x}(x)\}_{x\in X}$ is an open cover of the compact subset X; hence there exists $x_1, x_2, \ldots, x_m \in X$ such that $X = \bigcup_{j=1}^{m} B_{\delta_{x_j}}$.

which implies that, for all $x \in B_{\delta_{x_j}}$ and for all $i = 0, \ldots, n$

$$\|f^{(i)}(x) - f^{(i)}(x_j)\|_1 < \frac{\varepsilon}{4(n+1)} \text{ for all } f \in F.$$

Since F is equi-bounded, then the set

$\mathcal{F} = \{(f(x_j), f'(x_j), \ldots, f^{(n)}(x_j)), j = 1, \ldots, m; f \in F\}$ is bounded.

Since a bounded set in $\mathbb{R}^{n+1}$ is totally bounded, then there exists a subset $\{(y_{1,i}, y_{2,i}, \ldots, y_{n+1,i}), i = 1, \ldots, k\} \subset \mathbb{R}^{n+1}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{4(n+1)}}(y_{1,i}, y_{2,i}, \ldots, y_{n+1,i})$$

For any application φ: $\{1, \ldots, m\} \to \{1, \ldots, k\}$, we define the set

$$\mathcal{F}_\varphi = \left\{ f \in F : \left(f(x_j), f'(x_j), \ldots, f^{(n)}(x_j)\right) \in B_{\frac{\varepsilon}{4(n+1)}}\left(y_{1,\varphi_j}, y_{2,\varphi_j}, \ldots, y_{n+1,\varphi_j}\right), j = 1, \ldots, m \right\}.$$

It is clear that $F = \bigcup \mathcal{F}_\varphi$. Now, we show that the diameter of $\mathcal{F}_\varphi$ is less than ε.

Let $f, g \in \mathcal{F}_\varphi$ and $x \in X$, then there exists $j \in \{1, \ldots, m\}$ such that $x \in B_{\delta_{x_j}}$.

Hence, for all $i = 1, \ldots, n$

$$\begin{aligned} \|f^{(i)}(x) - g^{(i)}(x)\|_1 \leq \quad & \left\|f^{(i)}(x) - f^{(i)}(x_j)\right\|_1 + \left\|f^{(i)}(x_j) - y_{i+1,\varphi_j}\right\|_1 \\ & + \left\|g^{(i)}(x_j) - y_{i+1,\varphi_j}\right\|_1 + \left\|g^{(i)}(x_j) - g^{(i)}(x)\right\|_1 \leq \varepsilon. \end{aligned}$$

which implies that the diameter of $\mathcal{F}_\varphi$ is less than ε. Therefore, F can be covered by finitely many sets of diameter less than ε.

Thus F is totally bounded, and the proof is completed. □

3. Application to the solution of a higher-order boundary value problem

In this section, we study the existence of a solution for the problem (1.1).

It is easy to check, (see [1]), that u is a solution of (1.1) in $C^n(I, \mathbb{R})$ if and only if u is a solution of the following integro-differential equation:

$$u(t) = \int_0^1 G(t,s) f\left(s, u, u', \ldots, u^{(n-2)}\right) ds, \tag{3.1}$$

in $C^{n-2}(I, \mathbb{R})$, such that $g(t,s) = \frac{\partial^{n-2} G(t,s)}{\partial t^{n-2}}$ is the Green's function of the second-order boundary value problem

$$\begin{cases} -u^{(2)} = 0, t \in [0,1], \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

Moreover,

$$g(t,s) = \frac{1}{\alpha\gamma + \alpha\delta + \beta\gamma} \begin{cases} (\beta + \alpha s)[\delta + \gamma(1-t)], 0 \le s \le t, \\ (\beta + \alpha t)[\delta + \gamma(1-s)], t \le s \le 1. \end{cases} \tag{3.2}$$

Before stating our main result, we recall the following Schauder fixed point theorem.

Theorem 3.1. [14] *Let C be a non-empty, bounded, closed and convex subset of a Banach space E and T is a continuous operator from C into itself. If T(C) is relatively compact, then T has a fixed point.*

Equation (3.1) will be studied under the following assumptions:

[(*i*)] $f \in C(I \times \mathbb{R}^{n-1}, \mathbb{R})$.

[(*ii*)] There exists a function $a \in C(I, \mathbb{R}^+)$ and constants $b_k \in \mathbb{R}^+ (k = 0, \ldots, n-2)$ such that

$$|f(s, u_0, u_1, \ldots, u_{n-2})| \le a(s) + \sum_{k=0}^{n-2} b_k |u_k|$$

Under the assumptions (*i*) and (*ii*), we will make use of Schauder fixed point theorem to prove the following main result:

Theorem 3.2. *If the hypotheses* (*i*) *and* (*ii*) *hold, and if*

$$r \sum_{i=0}^{n-2} \left\| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \right\|_\infty < 1$$

such that $r = Max\{b_0, \ldots, b_{n-2}\}$.

Then, the integro-differential Equation (3.1) *has a solution in* $C^{n-2}(I, \mathbb{R})$.

Proof. Solving Equation (3.1) is equivalent to finding a fixed point of the operator A defined in the space $E = C^{n-2}(I, \mathbb{R})$ by the following expression:

$$Ax(t) = \int_0^1 G(t,s) f\left(s, x, x', \ldots, x^{(n-2)}\right) ds.$$

It is clear that the operator A is well defined from E into itself.

Moreover for all $x \in E$, $t \in I$ and $i = 0, \ldots, n-2$, we have

$$(Ax)^{(i)}(t) = \int_0^1 \partial_1^{(i)} G(t,s) f\left(s, x, x', \ldots, x^{(n-2)}\right) ds.$$

The proof is split into three steps.

Step I. There exists $\alpha > 0$ such that A transforms $C = \{x \in E, \|x\| \le \alpha\}$ into itself. It is clear that C is non-empty, bounded, closed and convex subset of E.

Moreover, for all $x \in C$, $t \in I$ and $i = 0, \ldots n-2$, we have

$$|(Ax)^{(i)}(t)| = \left| \int_0^1 \partial_1^{(i)} G(t,s) f\left(s, x, x', \ldots, x^{(n-2)}\right) ds \right|$$
$$\leq \int_0^1 |\partial_1^{(i)} G(t,s)| \left(a(s) + \sum_{k=0}^{n-2} b_k |x^{(k)}(s)| \right) ds \tag{3.3}$$
$$\leq \left(\|a\|_\infty + \sum_{k=0}^{n-2} b_k \|x^{(k)}\|_\infty \right) \int_0^1 |\partial_1^{(i)} G(t,s)| ds.$$

Hence, for $r = Max\{b_0, \ldots, b_{n-2}\}$, we obtain

$$\begin{aligned} \|Ax\| &= \sum_{i=0}^{n-2} \|A^{(i)} x\|_\infty \\ &\leq (\|a\|_\infty + r\alpha) \sum_{i=0}^{n-2} \left\| \int_0^1 \left|\partial_1^{(i)} G(t,s)\right| ds \right\|_\infty \end{aligned}$$

We deduce that, A transforms C into itself if

$$(\|a\|_\infty + r\alpha) \sum_{i=0}^{n-2} \left\| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \right\|_\infty \leq \alpha.$$

which implies, under the condition of Theorem (3.2), that

$$\frac{\left\| \|a\|_\infty \sum_{i=0}^{n-2} \left\| \int_0^1 \left|\partial_1^{(i)} G(t,s)\right| ds \right\| \right\|_\infty}{1 - r \sum_{i=0}^{n-2} \left\| \int_0^1 \left|\partial_1^{(i)} G(t,s)\right| ds \right\|_\infty} \leq \alpha.$$

Then, A transforms C into itself for

$$\alpha = \frac{\left\| \|a\|_\infty \sum_{i=0}^{n-2} \left\| \int_0^1 \left|\partial_1^{(i)} G(t,s)\right| ds \right\| \right\|_\infty}{1 - r \sum_{i=0}^{n-2} \left\| \int_0^1 \left|\partial_1^{(i)} G(t,s)\right| ds \right\|_\infty}.$$

Step 2: The operator A is continuous.

Let $(x_m) \in C$ be a convergence sequence to $x \in C$, which implies that $(x_m^{(i)})$ converges to $x^{(i)}$ in the space $C(I, [-\alpha, \alpha])$ for all $i = 0, \ldots, n-2$.

Since f is uniformly continuous on the compact set $I\times\underbrace{[-\alpha,\alpha]\times\cdots\times[-\alpha,\alpha]}_{n-1\text{ times}}$, then the sequence $(f(s,x_m,x'_m,\ldots,x_m^{(n-2)}))$ converges to $f(s,x,x',\ldots,x^{(n-2)})$ in $C(I,\mathbb{R})$.

It follows that

$$\|Ax_m-Ax\|_{\cdot}\leq\|f(s,x_m,x'_m,\ldots,x_m^{n-2})-f(s,x,x',\ldots,x^{n-2})\|_\infty\sum_{i=0}^{n-2}\|\int_0^1\partial_1^{(i)}G(t,s)ds\|_\infty.$$

which implies that (Ax_m) converges to Ax, and the operator A is continuous.

Step 3: $A(C)$ is relatively compact; it is clear that $A(C)$ is equi-bounded.

Now, to show that $A(C)$ is equi-continuous, take t_1 and t_2 in I.

Then, for all $i=0,\ldots n-3$, there exists ξ_i between t_1 and t_2 such that

$$\partial_1^{(i)}G(t_2,s)-\partial_1^{(i)}G(t_1,s)=(t_2-t_1)\partial_1^{(i+1)}G(\xi_i,s).$$

Hence, for all $i=0,\ldots n-3$,

$$\begin{aligned}|Ax^{(i)}(t_2)-Ax^{(i)}(t_1)|&=\left|\int_0^1 f(s,x,x',\ldots,x^{(n-2)})\left(\partial_1^{(i)}G(t_2,s)-\partial_1^{(i)}G(t_1,s)\right)ds\right|\\&\leq\int_0^1|f(s,x,x',\ldots,x^{(n-2)})\partial_1^{(i+1)}G(\xi_i,s)(t_2-t_1)|ds\\&\leq|t_2-t_1|(\|a\|_\infty+r\alpha)\left\|\int_0^1\left|\partial_1^{(i+1)}G(t,s)\right|ds\right\|_\infty\end{aligned}\tag{3.4}$$

Now, let $\varepsilon>0$. We note $\lambda=\max\limits_{0\leq i\leq n-3}\left\|\int_0^1\left|\partial_1^{(i+1)}G(t,s)\right|ds\right\|_\infty$.

Then from (3.4), if $|t_2-t_1|\leq\delta_1=\frac{\varepsilon}{1+(\|a\|_\infty+r\alpha)\lambda}$, we have for all $i=0,\ldots,n-3$,

$$|Ax^{(i)}(t_2)-Ax^{(i)}(t_1)|\leq\varepsilon$$

On the other hand, since the function $g(t,s)$ is uniformly continuous on $I\times I$, there exists $\delta_2>0$ such that if $|t_2-t_1|\leq\delta_2$, then for all $s\in I$

$$|g(t_2,s)-g(t_1,s)|<\frac{\varepsilon}{1+\|a\|_\infty+r\alpha}.$$

which implies, for $i=n-2$, that

$$\begin{aligned}|(Ax)^{(n-2)}x(t_2)-(Ax)^{(n-2)}x(t_1)|&=|\int_0^1 f(s,x,x',\ldots,x^{(n-2)})(g(t_2,s)-g(t_1,s))ds|\\&\leq(\|a\|_\infty+r\alpha)\|g(t_2,s)-g(t_1,s)\|_\infty\\&\leq\varepsilon.\end{aligned}$$

Hence, the third step is completed by setting $\delta=\min(\delta_1,\delta_2)$. Therefore, the set $A(C)$ is equi-continuous.

The proof of Theorem 3.2 then follows from Schauder fixed point theorem. □

Example 3.3. *Consider the following third-order boundary value problem:*

$$\begin{cases} u^{(3)} + \lambda \ln\left(2 + u^2 + (u')^2\right) = 0, t \in I = [0,1], \\ u(0) = 0, \\ u'(0) - u^{(2)}(0) = 0, \\ u'(1) + u^{(2)}(1) = 0. \end{cases} \tag{3.5}$$

where λ is a positive number. Hence, by using the notations and the parameters of Theorem 3.2,

$$n = 3, f(t,u,u') = \lambda \ln\left(2 + u^2 + (u')^2\right), \alpha = \gamma = \beta = \delta = 1, \frac{\partial G(t,s)}{\partial t} = g(t,s),$$

where,

$$g(t,s) = \frac{1}{3}\begin{cases} (1+s)[1+(1-t)], 0 \le s \le t, \\ (1+t)[1+(1-s)], t \le s \le 1. \end{cases}$$

which implies that $\int_0^1 |g(t,s)|ds = \frac{1}{2}(1 - t + t^2)$ and $\| \int_0^1 |g(t,s)|ds\| = \frac{5}{8}$.
On the other hand, we have

$$G(t,s) = \int_0^t g(r,s)dr = \frac{1}{3}\begin{cases} (1+s)\left[2t - \frac{t^2}{2}\right], 0 \le s \le t, \\ (2-s)\left[t + \frac{t^2}{2}\right], t \le s \le 1. \end{cases}$$

which implies that $\int_0^1 |G(t,s)|ds = \frac{1}{4}(2t + t^2)$ and $\| \int_0^1 |G(t,s)|ds\| = \frac{3}{4}$.

It is easy to see that $|f(s, u_0, u_1)| \le \lambda \ln(2) + \lambda|u_0| + \lambda|u_1|$.

Hence, the conditions (i) and (ii) are fulfilled with $a(s) = \lambda \ln(2)$, $b_0 = b_1 = \lambda$.

Therefore, the inequality in Theorem 3.2 takes the form

$$\lambda\left(\frac{5}{8} + \frac{1}{3}\right) < 1 \Leftrightarrow \lambda < \frac{8}{11}.$$

Then by Theorem 3.2, we conclude that the third-order boundary value problem (3.5) has a solution $u \in C^3(I, \mathbb{R})$ if $\lambda < \frac{8}{11}$.

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Corresponding author
Azzeddine Bellour can be contacted at: bellourazze123@yahoo.com