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Ultrametric Fredholm operators and approximate pseudospectrum

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Abstract

Purpose – The paper deals with ultrametric bounded Fredholm operators and approximate pseudospectra of closed and densely defined (resp. bounded) linear operators on ultrametric Banach spaces.
Design/methodology/approach – The author used the notions of ultrametric bounded Fredholm operators and approximate pseudospectra of operators.
Findings – The author established some results on ultrametric bounded Fredholm operators and approximate pseudospectra of closed and densely defined (resp. bounded) linear operators on ultrametric Banach spaces.
Originality/value – The results of the present manuscript are original.

Keywords Ultrametric Banach spaces, Linear operators, Fredholm operators, Approximate pseudospectrum
Paper type Research paper

1. Introduction

In ultrametric operator theory, Serre [1] studied the operator $I - A$ where A is a completely continuous linear operator on a free Banach space. On the other hand, Gurson [2] lifted this restriction by working on general ultrametric Banach spaces. Recently, Nadathur [3] extended and studied some classical results on compact and Fredholm operators on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Moreover, Schikhof gave a basic theory for compact and semi-Fredholm operators on ultrametric Banach spaces, for more details, we refer to Ref. [4]. Furthermore, Perez-Garcia [5] studied the Calkin algebras and semi-Fredholm operators on ultrametric Banach spaces. The stability of Fredholm operators and semi-Fredholom operators under smallest perturbation of operators and under compact operators on ultrametric Banach spaces were proved by Araujo, Perez-Garcia and Vega [6–8]. There are many studies on ultrametric Fredholm operators, see Refs. [1–3,5,6,8–10].

The pseudospectra of bounded linear operators and the pseudospectra of bounded linear operator pencils and the condition pseudospectra of matrices and bounded linear operators were extended and studied by several authors, see Refs. [11–14].

In this paper, we demonstrate some results on Fredholm operators on ultrametric Banach spaces. On the other hand, we introduce and study the approximate pseudospectra of closed and

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densely defined linear operators on ultrametric Banach spaces. In particular, we prove that the approximate pseudospectra associated with various ε are nested sets and that the intersection of all the approximate pseudospectra is the approximate spectrum. On the other hand, we introduce the essential approximate pseudospectra and we study some of its properties.

Throughout this paper, E, F and G are infinite-dimensional ultrametric Banach spaces over a complete ultrametric valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$, $\mathcal{L}(E,F)$ denotes the set of all continuous linear operators from E into F, I_E is the identity operator on E and I_F is the identity operator on F. If $E=F$, we have $\mathcal{L}(E,F)=\mathcal{L}(E)$. If $A\in\mathcal{L}(E)$, $N(A)$ and $R(A)$ denote the kernel and the range of A respectively. For more details see Refs. [9,15]. Recall that, an unbounded linear operator $A: D(A)\subseteq E\to F$ is said to be closed if for each $(x_n)\subset D(A)$ such that $\|x_n-x\|\to 0$ and $\|Ax_n-y\|\to 0$ as $n\to\infty$, for some $x\in E$ and $y\in F$, then $x\in D(A)$ and $y=Ax$. A is called densely defined if $D(A)$ is dense in E. The collection of closed linear operators from E into F is denoted by $\mathcal{C}(E,F)$. If $E=F$, we put $\mathcal{C}(E,F)=\mathcal{C}(E)$.

2. Preliminaries

We continue by recalling some preliminaries.

Definition 2.1. [7] We say that $A\in\mathcal{L}(E,F)$ has an index when both $\alpha(A)=\dim N(A)$ and $\beta(A)=\dim(F/R(A))$ are finite. In this case, the index of the linear operator A is defined as $ind(A)=\alpha(A)-\beta(A)$.

Definition 2.2. [7] Let $A\in\mathcal{L}(E,F)$. A is said to be an upper semi-Fredholm operator if $\alpha(A)$ is finite and $R(A)$ is closed. The set of all upper semi-Fredholm operators from E into F is denoted by $\Phi_+(E,F)$.

Definition 2.3. [7] Let $A\in\mathcal{L}(E,F)$, A is said to be a lower semi-Fredholm operator if $\beta(A)$ is finite. The set of all semi-Fredholm operators from E into F is denoted by $\Phi_-(E,F)$.
The set of all Fredholm operators from E into F is defined by

$$\Phi(E,F)=\Phi_+(E,F)\cap\Phi_-(E,F).$$

Definition 2.4. [15] Let E and F be two ultrametric Banach spaces over $\mathbb{K}$. A linear map $A: E\to F$ is said to be compact if $A(B_E)$ is compactoid in F, where $B_E=\{x\in E: \|x\|\leq 1\}$.
The collection of all compact operators from E into F is denoted by $\mathcal{K}(E,F)$.

Definition 2.5. [9] Let $A\in\mathcal{L}(E,F)$. A is called an operator of finite rank if $R(A)$ is a finite dimensional subspace of F.

Theorem 2.6. [15] Let $A\in\mathcal{L}(E,F)$. Then A is compact if and only if for each $\varepsilon>0$, there exists an operator $S\in\mathcal{L}(E,F)$ such that R(S) is finite-dimensional and $\|A-S\|<\varepsilon$.

Definition 2.7. [9] Let E be an ultrametric Banach space and let $S\in\mathcal{L}(E)$. S is said to be completely continuous if, there exists a sequence of finite rank linear operators (A_n) such that $\|A_n-S\|\to 0$ as $n\to\infty$.
The collection of all completely continuous linear operators on E is denoted by $\mathcal{C}_c(E)$.

Example 2.8. [9] Classical examples of completely continuous operators include finite rank operators.

Theorem 2.9. *[12] Suppose that $\mathbb{K}$ is spherically complete. Then, for each $A\in\Phi(E,F)$ and $K\in\mathcal{C}_c(E,F)$, $A+K\in\Phi(E,F)$ and $ind(A+K)=ind(A)$.*

Theorem 2.10. [16] Assume that E, F are ultrametric Banach spaces. Let A: D(A) $\subseteq$ E $\to$ F be a surjective closed linear operator. Then A is an open map.

Let $A: D(A) \subseteq E \to F$. When the domain of A is dense in E, the adjoint operator A' of A is defined as usual. Specifically, the operator $A': D(A') \subseteq F' \to E'$ satisfies

$$\langle Ax,\ y' \rangle = \langle x,\ A'y' \rangle$$

for all $x \in D(A)$, $y' \in D(A')$. As in the classical case, the following property is an immediate consequence of the definition.

Proposition 2.11. [16] Let A be a linear operator with dense domain. Then A′ is a closed linear operator.

Proposition 2.12. [16] Let A be a linear operator with dense domain. Then the following statement holds:

$$R(A)^{\perp} = N(A') \cong (F/\overline{R(A)})'.$$

For more details on bounded linear operators, see Ref. [17].

Theorem 2.13. [18] Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. For each $x \in E \backslash \{0\}$, there is $x' \in E'$ such that $x'(x) = 1$ and $\|x'\| = \|x\|^{-1}$.

Lemma 2.14. [19] Let E be an ultrametric normed vector space over a spherically complete field $\mathbb{K}$, and suppose that $E = N \oplus E_0$, where E_0 is a closed subspace and N is finite dimensional. If E_1 is a subspace of E containing E_0, then E_1 is closed.

Definition 2.15. [20] Let E and F be two ultrametric Banach spaces and let $A \in \mathcal{L}(E,F)$.

*(1).*The operator A is called Fredholm perturbation if $A + B \in \Phi(E, F)$ whenever $B \in \Phi(E, F)$.

(2).A is called an upper (resp. lower) semi-Fredholm perturbation $A + B \in \Phi_+(E, F)$ (resp. $A + B \in \Phi_-(E, F)$) whenever $B \in \Phi_+(E, F)$ (resp. $B \in \Phi_-(E, F)$).
We denote by $\mathcal{F}(E,F)$ the set of Fredholm perturbations and by $\mathcal{F}_+(E,F)$ (resp. $\mathcal{F}_-(E,F)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations. For $E = F$, we put $\mathcal{F}(E,F) = \mathcal{F}(E), \mathcal{F}_+(E,F) = \mathcal{F}_+(E)$ and $\mathcal{F}_-(E,F) = \mathcal{F}_-(E)$. The proof of the next proposition is similar to the classical case, see Ref. [20].

Proposition 2.16. [20] Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. (i) If $A \in \Phi(E)$ and $F \in \mathcal{F}(E)$, then $A + F \in \Phi(E)$ and $ind(A + F) = ind(A)$. (ii) If $A \in \Phi_+(E)$ and $F \in \mathcal{F}_+(E)$, then $A + F \in \Phi_+(E)$ and $ind(A + F) = ind(A)$.
The proof of the next theorem is similar to the classical case, see Ref. [20].

Theorem 2.17. [20] Let E be an ultrametric Banach space over $\mathbb{K}$. Let $A \in \Phi_+(E)$. Then the following statements are equivalent: (i) $ind(A) \leq 0$; (ii) A can be expressed in the form $A = S + K$ where $K \in \mathcal{C}_c(E)$, and $S \in \mathcal{C}(E)$ is an operator with closed range with $\alpha(S) = 0$.

Theorem 2.18. [3] Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Let $A \in \mathcal{L}(E,F)$. If there is $A_0, A_1 \in \mathcal{L}(F,E)$ such that $A_0A - I_E \in \mathcal{C}_c(E)$ and $AA_1 - I_F \in \mathcal{C}_c(F)$. Then $A \in \Phi(E, F)$.

Theorem 2.19. [3] Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Let $A \in \Phi(E, F)$, then there is $A_0 \in \mathcal{L}(F,E)$ such that $A_0A - I_E$ and $AA_0 - I_F$ have finite dimensional images.

Theorem 2.20. [3] *Let E, F and G be three ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *If* $A \in \Phi(E, F)$ *and* $B \in \Phi(F, G)$, *then* $BA \in \Phi(E, G)$ *and* $ind(BA) = ind(A) + ind(B)$.

Theorem 2.21. [3] *Let E be an ultrametric Banach space over a spherically complete field* $\mathbb{K}$. *Let* $A \in \mathcal{C}_c(E)$ *and* $\lambda \in \mathbb{K}\backslash\{0\}$, *then* $\lambda I_E - A \in \Phi(E)$ *and* $ind(\lambda I_E - A) = 0$.

Theorem 2.22. [9] *If* $A, B \in \mathcal{C}_c(E)$ *and* $C, D \in \mathcal{L}(E)$, *then (i)* $A + B \in \mathcal{C}_c(E)$; *(ii)* $AC, DA \in \mathcal{C}_c(E)$.

Lemma 2.23. [12] *Let E and F be two ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \mathcal{L}(E, F)$ *and* $K \in \mathcal{C}_c(E, F)$. *Then* $A + K \in \Phi(E, F)$ *and* $ind(A + K) = ind(A) + ind(K)$.

Corollary 2.24. [16] *Suppose that E, F are ultramtric Banach spaces. Let A be a closed linear operator with dense domain. If R(A) is a closed subspace which has the weak extension property in F, then* $R(A') = N(A)^{\perp}$.

In the next proposition, we assume that A' exists.

Proposition 2.25. [3] *Let E and F be two ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \Phi(E, F)$, *then* $A' \in \Phi(F', E')$ *and* $ind(A') = -ind(A)$.

3. Results

As a simple consequence of Theorems 2.18 and 2.20, we have:

Corollary 3.1. *Let E and F be two ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \Phi(E, F)$ *and let* $A_0 \in \mathcal{L}(F, E)$ *be such that* $A_0A - I_E$ *and* $AA_0 - I_F$ *are of finite rank. Then* $A_0 \in \Phi(F, E)$ *and* $ind(A_0) = -ind(A)$.

Proof. Since $A_0A - I_E$ and $AA_0 - I_F$ are of finite rank, we get $A_0A - I_E \in \mathcal{C}_c(E)$ and $AA_0 - I_F \in \mathcal{C}_c(F)$. Using Theorem 2.18, we have $A_0 \in \Phi(F, E)$. Since $A_0 \in \Phi(F, E)$ and $A \in \Phi(E, F)$. By Theorem 2.20, $A_0A \in \Phi(E)$ and $ind(A_0A) = ind(A) + ind(A_0)$. From Theorem 2.21, $ind(A_0A) = ind(A) + ind(A_0) = ind(I_E + B) = 0$, where $B = A_0A - I_E$ is of finite rank. □

Theorem 3.2. *Let E, F and G be three ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \mathcal{L}(E, F)$ *and* $B \in \mathcal{L}(F, G)$ *such that* $BA \in \Phi(E, G)$. *Then* $A \in \Phi(E, F)$ *if and only if* $B \in \Phi(F, G)$.

Proof. Suppose that $A \in \Phi(E, F)$. By Theorem 2.19, there is $A_0 \in \mathcal{L}(F, E)$ such that $A_0A - I_E$ and $AA_0 - I_F$ are of finite rank. By Theorem 2.18, $A_0 \in \Phi(F, E)$. Set $C = AA_0 - I_F$, then $BC = BAA_0 - B$. Since $BA \in \Phi(E, G)$. From Theorem 2.20, $BAA_0 \in \Phi(F, G)$. From Example 2.8, $C \in \mathcal{C}_c(F)$. By Theorem 2.22, we have $BC \in \mathcal{C}_c(F, G)$. From Lemma 2.23, $B \in \Phi(F, G)$. Similarly we obtain that if $B \in \Phi(F, G)$, hence $A \in \Phi(E, F)$. □

Theorem 3.3. *Let E, F and G be three ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \mathcal{L}(E, F)$ *and* $B \in \mathcal{L}(F, G)$ *be such that* $BA \in \Phi(E, G)$. *If* $\alpha(B)$ *is finite, then* $A \in \Phi(E, F)$ *and* $B \in \Phi(F, G)$.

Proof. Since $R(BA) \subset R(B)$, by Lemma 2.14, we get that $R(B)$ is closed. Since $R(BA) \subset R(B)$ and $\alpha(B)$ is finite, we have $\beta(B) \leq \beta(BA)$. Using the fact that $\alpha(B)$ is finite, we get $B \in \Phi(F, G)$. By Theorem 3.2, we have $A \in \Phi(E, F)$. □

Theorem 3.4. *Let E, F and G be three ultrametric Banach spaces over a spherically complete field* $\mathbb{K}$. *Let* $A \in \mathcal{L}(E, F)$ *and* $B \in \mathcal{L}(F, G)$ *be such that* $BA \in \Phi(E, G)$. *If* $\beta(A)$ *is finite, then* $A \in \Phi(E, F)$ *and* $B \in \Phi(F, G)$.

Proof. If $BA \in \Phi(E, G)$, then by Proposition 2.25, $A'B' \in \Phi(G', E')$. Since $\alpha(A') = \beta(A)$ is finite, from Theorem 3.3, we get $A' \in \Phi(F', E')$ and $B' \in \Phi(G', F')$. Furthermore, $\alpha(B'')$ is finite. Also

$$\alpha(B) \leq \alpha(B'') \text{ is finite.}$$

Using Theorem 3.3, we get $A \in \Phi(E, F)$ and $B \in \Phi(F, G)$. □

Lemma 3.5. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$*. Let* $A_1, \ldots, A_n \in \mathcal{L}(E)$ *be such that for all* $i, j \in \{1, \ldots, n\}$, $A_iA_j = A_jA_i$*. Suppose that* $A = A_1 \cdots A_n \in \Phi(E)$*. Then for all* $k \in \{1, \ldots, n\}$, $A_k \in \Phi(E)$*.*

Proof. One can see that for each $k \in \{1, \ldots, n\}$, $N(A_k) \subset N(A)$ and $R(A) \subset R(A_k)$. If $A = A_1 \cdots A_n \in \Phi(E)$, then for all $k \in \{1, \ldots, n\}$, $\alpha(A_k)$ and $\beta(A_k)$ are finite. Since $R(A)$ is closed and $\beta(B)$ is finite and $\mathbb{K}$ is spherically complete, then there is a finite-dimensional subspace M of E such that $E = R(A) \oplus M$. Since for any $k \in \{1, \ldots, n\}$, $R(A) \subset R(A_k)$, from Lemma 2.14, we have $R(A_k)$ is closed for each $k \in \{1, \ldots, n\}$. □

Theorem 3.6. Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$*. Then* $A \in \Phi(E, F)$*, if and only if* $A \in \Phi_+(E, F)$ *and* $A' \in \Phi_+(F', E')$*.*

Proof. From Proposition 2.25, if $A \in \Phi(E, F)$, then $A' \in \Phi(F', E')$. Thus $A \in \Phi_+(E, F)$ and $A' \in \Phi_+(F', E')$. Conversely, if $A \in \Phi_+(E, F)$ and $A' \in \Phi_+(F', E')$, then $\alpha(A)$ is finite and $R(A)$ is closed. From the fact that $A' \in \Phi_+(F', E')$, we get $\alpha(A') = \beta(A)$ is finite. Thus $A \in \Phi(E, F)$. □

We introduce the following definitions:

Definition 3.7. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(E)$ be closed and densely defined. The approximate spectrum $\sigma_{ap}(A)$ of A on E is defined by

$$\sigma_{ap}(A) = \left\{\lambda \in \mathbb{K} : \inf_{x \in D(A), \|x\|=1} \left\|(A - \lambda I_E)x\right\| = 0\right\}.$$

Definition 3.8. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. The approximate pseudospectrum $\sigma_{ap,\varepsilon}(A)$ of A on E is defined by

$$\sigma_{ap,\varepsilon}(A) = \sigma_{ap}(A) \cup \left\{\lambda \in \mathbb{K} : \inf_{x \in D(A), \|x\|=1} \left\|(A - \lambda I_E)x\right\| < \varepsilon\right\}.$$

Proposition 3.9. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$*. Let* $A \in \mathcal{C}(E)$ *be closed and densely defined. Then, the following statements hold:*

(1). For any $\varepsilon > 0$*, we have* $\sigma_{ap,\varepsilon}(A) \subseteq \sigma_\varepsilon(A)$*;*

(2). $\sigma_{ap}(A) = \bigcap_{\varepsilon > 0} \sigma_{ap,\varepsilon}(A)$*;*

(3). For all ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$*, we have* $\sigma_{ap}(A) \subset \sigma_{ap,\varepsilon_1}(A) \subset \sigma_{ap,\varepsilon_2}(A)$*;*

(4). For all $\mu \in \mathbb{K}$ *and* $\varepsilon > 0$*, we have* $\sigma_{ap,\varepsilon}(A + \mu I_E) = \sigma_{ap,\varepsilon} + \mu$*;*

(5). For each $\mu \in \mathbb{K} \backslash \{0\}$ *and* $\varepsilon > 0$*, we have* $\sigma_{ap,|\mu|\varepsilon}(\mu A) = \mu \sigma_{ap,\varepsilon}(A)$*.*

Proof.

(1). Let $\lambda \notin \sigma_\varepsilon(A)$. Then $\|(A - \lambda I_E)^{-1}\| \leq \varepsilon^{-1}$. On the other hand,

$$\frac{1}{\inf_{x\in D(A),\|x\|=1}\|(A-\lambda I_E)x\|} = \sup_{x\in D(A),\|x\|=1}\left(\frac{\|x\|}{\|(A-\lambda I_E)x\|}\right), = \sup_{x\in D(A)\setminus\{0\}}\left(\frac{\|x\|}{\|(A-\lambda I_E)x\|}\right), = \sup_{u\in E\setminus\{0\}}\left(\frac{\left\|(A-\lambda I_E)^{-1}u\right\|}{\|u\|}\right), = \left\|(A-\lambda I_E)^{-1}\right\| \le \varepsilon^{-1}.$$

Thus $\lambda \notin \sigma_{ap,\varepsilon}(A)$.

(2). From Definition 3.8, for each $\varepsilon > 0$, we see that $\sigma_{ap}(A) \subseteq \sigma_{ap,\varepsilon}(A)$. Thus $\sigma_{ap}(A) \subseteq \bigcap_{\varepsilon>0}\sigma_{ap,\varepsilon}(A)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0}\sigma_{ap,\varepsilon}(A)$, then for each $\varepsilon > 0$, $\lambda \in \sigma_{ap,\varepsilon}(A)$. If $\lambda \in \sigma_{ap}(A)$, there is nothing to prove. If $\lambda \notin \sigma_{ap}(A)$, then $\lambda \in \{\lambda \in \mathbb{K} : \inf_{x\in D(A),\|x\|=1}\|(A-\lambda I_E)x\| < \varepsilon\}$. Taking the limit as $\varepsilon \to 0^+$, we get $\inf_{x\in D(A),\|x\|=1}\|(A-\lambda I_E)x\| = 0$, which is a contradiction. Hence $\lambda \in \sigma_{ap}(A)$.

(3). Let ε_1 and ε_2 be such that $0 < \varepsilon_1 < \varepsilon_2$. If $\lambda \in \sigma_{ap,\varepsilon_1}(A)$, then $\inf_{x\in D(A),\|x\|=1}\|(A-\lambda I_E)x\| < \varepsilon_1 < \varepsilon_2$. Thus $\lambda \in \sigma_{ap,\varepsilon_2}(A)$.

(4). If $\lambda \in \sigma_{ap,\varepsilon}(A+\mu I_E)$, then either $\lambda \in \sigma_{ap}(A+\mu I_E)$ or $\inf_{x\in D(A),\|x\|=1}\|(A-(\lambda-\mu)I_E)x\| < \varepsilon$. Thus $\lambda \in \mu + \sigma_{ap,\varepsilon}(A)$. Similarly, if $\lambda \in \mu + \sigma_{ap,\varepsilon}(A)$, then $\lambda \in \sigma_{ap,\varepsilon}(A+\mu I_E)$.

(5). If $\lambda \in \sigma_{ap,|\mu|\varepsilon}(\mu A)$, then

$$\inf_{x\in D(A),\|x\|=1}\|\cdot(\mu A-\lambda I_E)x\| = \inf_{x\in D(A),\|x\|=1}\left\|(\mu A-\lambda I_E)x\right\|, = |\mu|\inf_{x\in D(A),\|x\|=1}\left\|\left(A-\frac{\lambda}{\mu}I_E\right)x\right\|, < |\mu|\varepsilon.$$

Thus $\frac{\lambda}{\mu} \in \sigma_{ap,\varepsilon}(A)$. Then $\sigma_{ap,|\mu|\varepsilon}(\mu A) \subseteq \mu\sigma_{ap,\varepsilon}(A)$. Similarly we get $\mu\sigma_{ap,\varepsilon}(A) \subseteq \sigma_{ap,|\mu|\varepsilon}(\mu A)$.

Hence $\sigma_{ap,|\mu|\varepsilon}(\mu A) = \mu\sigma_{ap,\varepsilon}(A)$. □

Theorem 3.10. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$, and let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. Then,

$$\sigma_{ap,\varepsilon}(A) = \bigcup_{S\in\mathcal{L}(E):\|S\|<\varepsilon}\sigma_{ap}(A+S).$$

Proof. Let $\lambda \in \bigcup_{S\in\mathcal{L}(E):\|S\|<\varepsilon}\sigma_{ap}(A+S)$. Then $\inf_{x\in D(A),\|x\|=1}\|(A+S-\lambda I_E)x\| = 0$. We will prove that $\lambda \in \sigma_{ap,\varepsilon}(A)$. From the estimate

$$\|(A-\lambda I_E)u\| = \|(A+S-\lambda I_E)u - Su\| \leq \max\{\|(A+S-\lambda I_E)u\|, \|Su\|\},$$

We infer that $\inf_{x\in D(A),\|x\|=1}\|(A-\lambda I_E)x\| < \varepsilon$. Conversely, suppose that $\lambda \in \sigma_{ap,\varepsilon}(A)$. We discuss two cases. First case: If $\lambda \in \sigma_{ap}(A)$, then it suffices to put $S = 0$. Second case: If $\lambda \notin \sigma_{ap}(A)$, then there is $y \in E\backslash\{0\}$ such that $\|y\| = 1$ and $\|(A-\lambda I_E)y\| < \varepsilon$. By Theorem 2.13, there exists $\phi \in E'$ such that $\phi(y) = 1$ and $\|\phi\| = \|y\|^{-1} = 1$. Define S on E by

$$Sx = -\phi(x)(A-\lambda I_E)y \quad \text{for all } x \in E.$$

Then, S is linear and

$$\|Sx\| = \|\phi(x)\|\|(A-\lambda I_E)y\| < \varepsilon\|x\|.$$

Then, $\|S\| < \varepsilon$. Furthermore, $\inf_{x\in D(A),\|x\|=1}\|(A+S-\lambda I_E)x\| = 0$, because

$$\inf_{x\in D(A),\|x\|=1}\|(A+S-\lambda I_E)x\| \leq \|(A+S-\lambda I_E)y\| \leq \|(A-\lambda I_E)y - \phi(y)(A-\lambda I_E)y\| = 0,$$

for $y \in E$.

Theorem 3.11. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. Let $S \in \mathcal{L}(E)$ be such that $\|S\| < \varepsilon$. Then,

$$\sigma_{ap,\varepsilon-\|S\|}(A) \subseteq \sigma_{ap,\varepsilon}(A+S) \subseteq \sigma_{ap,\varepsilon+\|S\|}(A).$$

Proof. If $\lambda \in \sigma_{ap,\varepsilon-\|S\|}(A)$, then by Theorem 3.10, there is $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon - \|S\|$ and

$$\lambda \in \sigma_{ap}(A+B) = \sigma_{ap}((A+S)+(B-S)).$$

Since $\|B-S\| \leq \|B\| + \|S\| < \varepsilon$, by Theorem 3.10, we get $\lambda \in \sigma_{ap,\varepsilon}(A+S)$. Similarly, if $\lambda \in \sigma_{ap,\varepsilon}(A+S)$, we obtain that $\lambda \in \sigma_{ap,\varepsilon+\|S\|}(A)$. □

Definition 3.12. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$, and let $A \in \mathcal{C}(E)$ be closed and densely defined. Then, the essential approximate spectrum $\sigma_{eap}(A)$ of A is defined by

$$\sigma_{eap}(A) = \bigcap_{C\in\mathcal{C}_c(E)} \sigma_{ap}(A+C).$$

Definition 3.13. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$, and let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. Then the essential approximate pseudospectrum $\sigma_{eap}(A)$ of A is defined by

$$\sigma_{eap,\varepsilon}(A) = \bigcap_{C \in \mathcal{C}_c(E)} \sigma_{ap,\varepsilon}(A + C).$$

Proposition 3.14. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$. *Let* $A \in \mathcal{C}(E)$ *be closed and densely defined. Then, the following statements hold:*

(1). $\sigma_{eap}(A) = \bigcap_{\varepsilon > 0} \sigma_{eap,\varepsilon}(A)$;

(2). For all ε_1 *and* ε_2 *such that* $\varepsilon_1 < \varepsilon_2$, *we have* $\sigma_{eap}(A) \subset \sigma_{eap,\varepsilon_1}(A) \subset \sigma_{eap,\varepsilon_2}(A)$;

(3). $\sigma_{eap,\varepsilon}(A + C) = \sigma_{eap,\varepsilon}(A)$, *for each* $C \in \mathcal{C}_c(E)$.

Proof.

(1). Let $\lambda \notin \sigma_{eap,\varepsilon}(A)$. Then, there is $C \in \mathcal{C}_c(E)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(A + C)$. Hence $\lambda \notin \sigma_{eap}(A)$. Thus $\sigma_{eap}(A) \subset \bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(A)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(A)$, then for each $\varepsilon > 0$, $\lambda \in \sigma_{eap,\varepsilon}(A)$. Hence, for all $C \in \mathcal{C}_c(E)$ such that $\lambda \in \sigma_{ap,\varepsilon}(A + C)$. Thus, $\inf_{x \in D(A), \|x\|=1} \|(A + C - \lambda I_E)x\| < \varepsilon$. Taking the limit as $\varepsilon \to 0$, we get $\inf_{x \in D(A), \|x\|=1} \|(A + C - \lambda I_E)x\| = 0$. Hence $\lambda \in \sigma_{eap}(A)$.

(2). If $\lambda \in \sigma_{eap,\varepsilon_1}(A)$, then for all $C \in \mathcal{C}_c(E)$ such that $\inf_{x \in D(A), \|x\|=1} \|(A + C - \lambda I_E)x\| < \varepsilon_1 < \varepsilon_2$. Thus $\lambda \in \sigma_{eap,\varepsilon_2}(A)$.

(3). Follow from Definition 3.13.

In the next theorem, we give a characterization of the essential approximate pseudospectrum by means of ultrametric semi-Fredholm operators.

Theorem 3.15. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$. *Let* $A \in \mathcal{C}(E)$ *be closed and densely defined, and let* $\varepsilon > 0$. *Then* $\lambda \notin \sigma_{eap,\varepsilon}(A)$ *if and only if for each* $B \in \mathcal{L}(E)$ *such that* $\|B\| < \varepsilon$, *we have* $A + B - \lambda I_E \in \Phi_+(E)$ *and* $ind(A + B - \lambda I_E) \leq 0$.

Proof. If $\lambda \notin \sigma_{eap,\varepsilon}(A)$, then there is $K \in \mathcal{C}_c(E)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(A + K)$. Using Theorem 3.10, for each $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon$, we have $\lambda \notin \sigma_{ap}(A + K + B)$. Hence for each $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon$, we obtain

$$A + K + B - \lambda I_E \in \Phi_+(E)$$

and

$$ind(A + K + B - \lambda I_E) \leq 0.$$

From Theorem 2.16, we have

$$A + B - \lambda I_E \in \Phi_+(E)$$

and

$$ind(A + B - \lambda I_E) \leq 0.$$

Conversely, suppose that for each $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon$, we have $A + B - \lambda I_E \in \Phi_+(E)$ and $ind(A + B - \lambda I_E) \leq 0$. Then from Theorem 2.17, we get

$$A + B - \lambda I_E = -(C + K),$$

where $K \in \mathcal{C}_c(E)$ and $C \in \mathcal{C}(E)$ with closed range $\alpha(C) = 0$. Hence

$$A + K + B - \lambda I_E = -C \tag{3.1}$$

and

$$ind(A + K + B - \lambda I_E) = \alpha(C) = 0.$$

Since C has a closed range and $\alpha(C) = 0$, by (3.1), there is $M > 0$ such that

$$\|(A + K + B - \lambda I_E)x\| \geq M\|x\|, \text{ for each } x \in D(A).$$

Hence $\inf_{x \in D(A), \|x\|=1} \|(A + K + B - \lambda I_E)x\| \geq M > 0$. Thus $\lambda \notin \sigma_{ap}(A + K + B)$. Consequently, $\lambda \notin \sigma_{eap,\varepsilon}(A)$. □

Remark 3.16. From Theorem 3.15, we get

$$\sigma_{eap,\varepsilon}(A) = \bigcup_{B \in \mathcal{L}(E): \|B\| < \varepsilon} \sigma_{eap}(A + B).$$

From Definition 3.13 and from Proposition 3.14, we have.

Corollary 3.17. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. Then, we have

$$\sigma_{eap}(A) = \lim_{\varepsilon \to 0} \overline{\bigcap_{C \in \mathcal{C}_c(E)} \sigma_{ap,\varepsilon}(A + C)} = \bigcap_{\varepsilon > 0} \left(\bigcup_{B \in \mathcal{L}(E): \|B\| < \varepsilon} \sigma_{eap}(A + B) \right).$$

Theorem 3.18. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon > 0$. Then, we have

$$\sigma_{eap,\varepsilon}(A) = \bigcap_{F \in \mathcal{F}_+(E)} \sigma_{ap,\varepsilon}(A + F).$$

Proof. If $\lambda \notin \bigcap_{F \in \mathcal{F}_+(E)} \sigma_{ap,\varepsilon}(A + F)$, then there is $F \in \mathcal{F}_+(E)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(A + F)$. By Theorem 3.10, for each $B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$, we have $\lambda \notin \sigma_{ap}(A + F + B)$. Hence, for each $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon$, we have

$$A + F + B - \lambda I_E \in \Phi_+(E),$$

and

$$ind(A+F+B-\lambda I_E)\leq 0.$$

From Theorem 2.16, we have

$$A+B-\lambda I_E\in\Phi_+(E)$$

and

$$ind(A+B-\lambda I_E)\leq 0.$$

By Theorem 3.15, we see that $\lambda\notin\sigma_{eap,\varepsilon}(A)$. Conversely, from $C_c(E)\subset\mathcal{F}_+(E)$, we infer that

$$\bigcap_{F\in\mathcal{F}_+(E)}\sigma_{ap,\varepsilon}(A+F)\subset\bigcap_{F\in\mathcal{C}_c(E)}\sigma_{ap,\varepsilon}(A+F)=\sigma_{eap,\varepsilon}(A).$$

Remark 3.19. *(i) From* Theorem 3.18, we have $\sigma_{eap,\varepsilon}(A+C)=\sigma_{eap,\varepsilon}(A)$, for each $C\in\mathcal{F}_+(E)$. (ii) Let $J(E)$ be a subset of $\mathcal{L}(E)$. If $\mathcal{C}_c(E)\subset J(E)\subset\mathcal{F}_+(E)$, then we have

$$\sigma_{eap,\varepsilon}(A)=\bigcap_{C\in J(E)}\sigma_{ap,\varepsilon}(A+C),$$

and

$$\sigma_{eap,\varepsilon}(A+C)=\sigma_{eap,\varepsilon}(A)\text{ for each }C\in J(E).$$

Theorem 3.20. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\|\subseteq|\mathbb{K}|$. Let $A\in\mathcal{C}(E)$ be closed and densely defined, and let $\varepsilon>0$. Let $S\in\mathcal{L}(E)$ be such that $\|S\|<\varepsilon$. Then, we have (i) $\sigma_{eap,\varepsilon-\|S\|}(A)\subseteq\sigma_{eap,\varepsilon}(A+S)\subseteq\sigma_{eap,\varepsilon+\|S\|}(A)$; (ii) For any $\lambda,\mu\in\mathbb{K}$ and $\mu\neq 0$, we have

$$\sigma_{eap,\varepsilon}(\lambda I_E+\mu A)=\lambda+\mu\sigma_{eap,\varepsilon|\mu|}(A).$$

Proof. Follow from Theorem 3.11 and Proposition 3.9. □

4. Bounded cases

First, we introduce the following definitions.

Definition 4.1. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\|\subseteq|\mathbb{K}|$. Let $A\in\mathcal{L}(E)$. The approximate spectrum $\sigma_{ap}(A)$ of A on E is defined by

$$\sigma_{ap}(A)=\left\{\lambda\in\mathbb{K}:\inf_{x\in D(A),\|x\|=1}\left\|(A-\lambda I_E)x\right\|=0\right\}.$$

Definition 4.2. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ be such that $\|E\|\subseteq|\mathbb{K}|$. Let $A\in\mathcal{L}(E)$ and $\varepsilon>0$. The approximate pseudospectrum $\sigma_{ap,\varepsilon}(A)$ of A on E is defined by

$$\sigma_{ap,\varepsilon}(A) = \sigma_{ap}(A) \cup \left\{ \lambda \in \mathbb{K} : \inf_{x \in D(A), \|x\|=1} \left\| (A - \lambda I_E)x \right\| < \varepsilon \right\}.$$

As a particular case of Proposition 3.9, we have:

Proposition 4.3. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$ *and let* $A \in \mathcal{L}(E)$. *Then, the following statements hold:*

(1). For any $\varepsilon > 0$, *we have* $\sigma_{ap,\varepsilon}(A) \subseteq \sigma_\varepsilon(A)$.

(2). $\sigma_{ap}(A) = \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(A)$;

(3). For all ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, *we have* $\sigma_{ap}(A) \subset \sigma_{ap,\varepsilon_1}(A) \subset \sigma_{ap,\varepsilon_2}(A)$;

(4). For all $\mu \in \mathbb{K}$ *and* $\varepsilon > 0$, *we have* $\sigma_{ap,\varepsilon}(A + \mu I_E) = \sigma_{ap,\varepsilon} + \mu$;

(5). For each $\mu \in \mathbb{K} \backslash \{0\}$ *and* $\varepsilon > 0$, *we have* $\sigma_{ap,|\mu|\varepsilon}(\mu A) = \mu \sigma_{ap,\varepsilon}(A)$.

As a particular case of Theorem 3.10, we have:

Theorem 4.4. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$, *and let* $A \in \mathcal{L}(E)$ *and* $\varepsilon > 0$. *Then, we have*

$$\sigma_{ap,\varepsilon}(A) = \bigcup_{S \in \mathcal{L}(E) : \|S\| < \varepsilon} \sigma_{ap}(A + S).$$

As a particular case of Theorem 3.11, we have:

Theorem 4.5. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ *such that* $\|E\| \subseteq |\mathbb{K}|$. *Let* $A, S \in \mathcal{L}(E)$ *be such that* $\|S\| < \varepsilon$. *Then, we have*

$$\sigma_{ap,\varepsilon - \|S\|}(A) \subseteq \sigma_{ap,\varepsilon}(A + S) \subseteq \sigma_{ap,\varepsilon + \|S\|}(A).$$

Definition 4.6. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$ and let $A \in \mathcal{L}(E)$. The essential approximate spectrum $\sigma_{eap}(A)$ of A is defined by

$$\sigma_{eap}(A) = \bigcap_{C \in \mathcal{C}_c(E)} \sigma_{ap}(A + C).$$

As a particular case of Definition 3.13, we have:

Definition 4.7. Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$, let $A \in \mathcal{L}(E)$ and $\varepsilon > 0$. The essential approximate pseudospectrum $\sigma_{eap}(A)$ of A is defined by

$$\sigma_{eap,\varepsilon}(A) = \bigcap_{C \in \mathcal{C}_c(E)} \sigma_{ap,\varepsilon}(A + C).$$

As a particular case of Proposition 3.14, we have:

Proposition 4.8. *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{L}(E)$. Then, the following statements hold:*

(1). $\sigma_{eap}(A) = \bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(A)$.

(2). For all ε_1 and ε_2 such that $\varepsilon_1 < \varepsilon_2$, we have $\sigma_{eap}(A) \subset \sigma_{eap,\varepsilon_1}(A) \subset \sigma_{eap,\varepsilon_2}(A)$.

(3). $\sigma_{eap,\varepsilon}(A + C) = \sigma_{eap,\varepsilon}(A)$, *for each $C \in \mathcal{C}_c(E)$.*

In the next theorem, we give a characterization of the essential approximate pseudospectra of bounded linear operators by means of ultrametric semi-Fredholm operators.

Theorem 4.9. *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then $\lambda \notin \sigma_{eap,\varepsilon}(A)$ if and only if for each $B \in \mathcal{L}(E)$ such that $\|B\| < \varepsilon$, we have $A + B - \lambda I_E \in \Phi_+(E)$ and $ind(A + B - \lambda I_E) \leq 0$.*

Proof. It is a particular case of Theorem 3.15. □

Remark 4.10. *From* Theorem 4.9, we get

$$\sigma_{eap,\varepsilon}(A) = \bigcup_{B \in \mathcal{L}(E):\|B\|<\varepsilon} \sigma_{eap}(A + B).$$

From Proposition 4.8, we have.

Corollary 4.11. *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then, we have*

$$\sigma_{eap}(A) = \lim_{\varepsilon \to 0} \overline{\bigcap_{C \in \mathcal{C}_c(E)} \sigma_{ap,\varepsilon}(A + C)} = \bigcap_{\varepsilon>0} \left(\bigcup_{B \in \mathcal{L}(E):\|B\|<\varepsilon} \sigma_{eap}(A + B) \right).$$

Theorem 4.12. *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then, we have*

$$\sigma_{eap,\varepsilon}(A) = \bigcap_{F \in \mathcal{F}_+(E)} \sigma_{ap,\varepsilon}(A + F).$$

Remark 4.13. (i) From Theorem 4.12, $\sigma_{eap,\varepsilon}(A + C) = \sigma_{eap,\varepsilon}(A)$, for each $C \in \mathcal{F}_+(E)$. (ii) Let $J(E)$ be a subset of $\mathcal{L}(E)$. If $\mathcal{C}_c(E) \subset J(E) \subset \mathcal{F}_+(E)$, then we have

$$\sigma_{eap,\varepsilon}(A) = \bigcap_{C \in J(E)} \sigma_{ap,\varepsilon}(A + C)$$

and

$$\sigma_{eap,\varepsilon}(A + C) = \sigma_{eap,\varepsilon}(A) \text{ for each } C \in J(E).$$

As a particular case of Theorem 3.20, we have:

Theorem 4.14. *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $A, S \in \mathcal{L}(E)$ and $\varepsilon > 0$ be such that $\|S\| < \varepsilon$. Then, we have (i) $\sigma_{eap,\varepsilon-\|S\|}(A) \subseteq \sigma_{eap,\varepsilon}(A+S) \subseteq \sigma_{eap,\varepsilon+\|S\|}(A)$; (ii) For any $\lambda, \mu \in \mathbb{K}$ and $\mu \neq 0$, we have*

$$\sigma_{eap,\varepsilon}(\lambda I_E + \mu A) = \lambda + \mu\sigma_{eap,\varepsilon|\mu|}(A).$$

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Further reading

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